Variational Formulation of Multi-Boundary Non-homogeneous Electrostatic Problems Galerkin's Method Approach

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Abstract— **The paper presents the application of certain elements of the single and double layer potential theory to boundary problems of electrostatics. These problems were formulated with the second order Fredholm integral equations.**

$$\sigma(x) - \lambda \oint_S \sigma(y)\mathcal{K}(x-y)dy = f(x)$$

The presented theory was illustrated with an example of three dielectrics with the volume electric load density. On the basis of the potentiality condition of the Fredholm operator an energy functional was constructed using the Weinberg method. There was given an alternate formulation of the problem based on the Galerkin method.

Keywords— **single layer potential, Fredholm equation system of the second kind, Galerkin variational identities, Wainberg energy functional**

I. Introduction

The integral equation method based on the single and double layer potential theory was introduced by Giunter [2] and extended to the elecro- and magnetostatic applications in the Lvov research area by Tozoni [6]. The current paper educes a second-order Fredholm system of equations with a weakly singular nucleus [5], [8] by using a single layer potential theory. The system describes the density distribution of secondary sources on the boundary surfaces of a non-homogeneous system of dielectrics. The obtained equation system was formulated as two variational identities in Galerkin sense [1], [4]. By using the Weinberg's theory of potentiality operators [7] there was given an alternative formulation of the considered electroctatics boundary problem in the form of an extreme energy functional question. The presented variational formulation of the integral system once as a variational identity in the Galerkin sense and then as an extreme energy functional question is part of the contemporary mathematical modeling of the boundary problems with distributed parameters. It constitutes a natural approach for approximation with the boundary finite element known in the literature as the Boundary Element Method BEM.

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II. Multi-layer electostatics problem

Let's now consider a model boundary problem of determining the $\varphi(Q)$ potential distribution in an unlimited space $\mathbb{R}^3$ filled with a system of dielectrics in which volume electric load densities exist. We shall consider a case where the outside of the solid Ω_o limited with a surface $\left\{\mathbb{R}^3\text{-}\Omega_o^i\right\}$ is filled with a homogeneous dielectric ϵ^e. Inside the solid Ω_o there are contained m solids Ω_k^i limited with surfaces S_k and filled with dielectics of permeability ϵ_k^i each. These dielectrics contain source electric load of density $\rho_k^i, k = 1, 2, ..., m$. The space between the solids $\{\Omega_o - \bigcup_{i=1}^m\}$ is filled with a dielectric ϵ_o^i. For such an electrostatic system we will formulate an appropriate differential boundary problem.

A. *Differential boundary problem*

(Question 1): For the given densities ${\rho_k}^e(Q)$ and ${\rho_k}^i(Q)$ where $Q \in \Omega_k^e$, $Q \in \Omega_k^i \subset \mathbb{R}^3$ and given dielectric permeabilities $\varepsilon_k^e, \varepsilon_k^i$ determine the distribution of potential $\varphi(Q)$

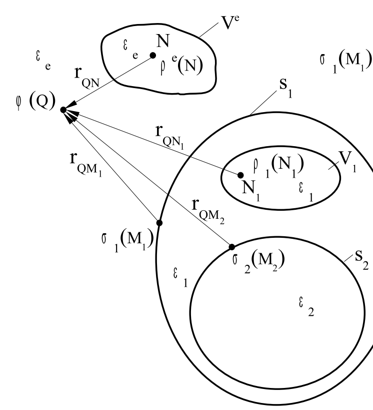

Fig. 1. *Non-homogeneous system of dielectrics and sources*

$$\varphi(x) = \begin{cases} \varphi^i(Q), & Q \in \Omega_k^i, k = 0, 1, 2, ..., m \\ \varphi^e(Q), & Q \in \left\{\mathbb{R}^3 - \Omega_0^i\right\} \end{cases} \tag{1}$$

that fulfills the following system of equations:

$$\nabla^2\varphi^i(x) = \begin{cases} \dfrac{-\rho^i(Q)_k}{\epsilon_k^i}, Q \in \left\{\Omega_k^i\right\}, k = 0, 1, ..., m \\ 0, \qquad Q \in \left\{\Omega_o^i - \bigcup_{k=1}^n \Omega_k^i\right\} \end{cases} \tag{2}$$

$$\nabla^2 \varphi^e(x) = \begin{cases} \dfrac{-\rho^e(Q)_k}{\epsilon_k^e}, & Q \in \{\Omega_k^e\}, k = 0,1,2,...,m \\ 0, & Q \in \{\mathbb{R}^3 - \bigcup_{k=1}^n \Omega_k^e\} \end{cases} \tag{3}$$

and where on the boundary surfaces S_k between the dielectrics ϵ^e i ϵ^i the boundary conditions are:

$$\varphi^i Q = \varphi^e(Q), \quad Q \in S_k \tag{4}$$

$$\epsilon^i \frac{\partial \varphi^i(Q)}{\partial \vec{n}} - \epsilon^e \frac{\partial \varphi^e(Q)}{\partial \vec{n}} = 0, \quad Q \in S_k \tag{5}$$

We shall reduce the formulated differantial-boundary problem to an integral problem by using the previously introduced potential theory.

B. *Integral boundary problem*

The single and double layer potential theory is given mathematically for a homogeneous material with material parameters equal to one. In order to meet the real permeability of the dielectrics system in the area Ω_k we shall transform the non-homogeneous system to a homogeneous one.

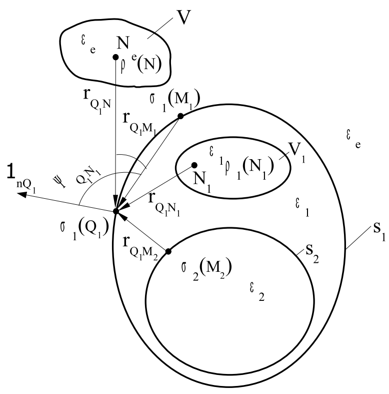

Fig. 2. *Induction scheme for the load on the boundary surface* S_1

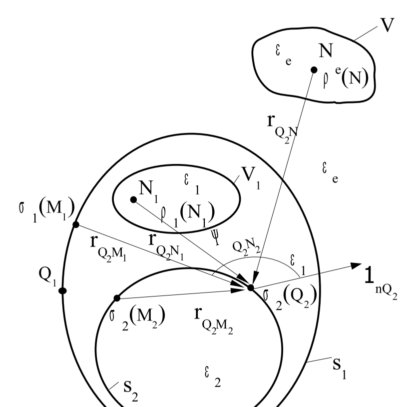

Fig. 3. *Induction scheme for the load on the boundary surface* S_2

$$\Psi : \{\epsilon_k^i\} \longrightarrow \{\epsilon^e\} \tag{6}$$

This transformation can be performed according to two different criteria of equivalence for the real and transformed system. Once by saving the electric field intensity vector ($\vec{E} = -\nabla \otimes \varphi = idem$) and then by saving the electric induction vector ($\vec{D} = idem$) in the areas where the dielectic has been shifted.

$$\vec{E} = -\nabla \otimes \varphi = idem \tag{7}$$

$$\vec{D} = idem \tag{8}$$

If we perform the medium transformation according to (7) then there will appear secondary sources on the boundary surfaces S_k. These sources will have the single layer potential density $\sigma_k(Q), Q \in S_k$. When shifting the dielectrics according to (8) then there will appear double layer surface potential densities $\tau_k(Q)$. The original volume sources are transformed according to the following rule:

$$\epsilon_k^i \rho_k^{i,}(Q) = \epsilon^e \rho_k^i(Q) \rightarrow \rho^{i,}{}_k(Q) = \frac{\epsilon^e}{\epsilon_k^i} \rho_k^i(Q) \tag{9}$$

Potential distribution in a homogeneous electrostatic system after transformation (6) is expressed in dielectric σ_e by the solution of the differential-boundary problem (Question 1 - (2)-(5)). This solution is in turn expressed by the single layer potential $\sigma_k(Q)$, reduced to volume potentials $\rho^{i,}{}_k(Q)$ and it receives the form:

$$\varphi(Q) = \frac{1}{4\pi\epsilon^e} \left\{ \oint_{S_1} \frac{\sigma_1(M_1)}{r_{QM_1}} dS_{M_1} + \oint_{S_2} \frac{\sigma_2(M_2)}{r_{QM_2}} dS_{M_2} \right\} +$$

$$+ \frac{1}{4\pi\epsilon^e} \left\{ \int_{\Omega_1} \frac{\rho_1^,(N_1)}{r_{QN_1}} dV_{N_1} + \int_{\Omega_e} \frac{\rho^e(N)}{r_{QN}} dV \right\} \tag{10}$$

For the potential distribution (10) to be a solution of the differential problem (Question 1) it has to conform on the boundary surfaces S_k to condition (5). This condition is equivalent in our case to the following system:

$$\epsilon^e \frac{\partial \varphi^+(Q_1)}{\partial \vec{n}} - \epsilon_1 \frac{\partial \varphi^-(Q_1)}{\partial \vec{n}} = 0, \quad Q_1 \in S_1 \tag{11}$$

$$\epsilon^e \frac{\partial \varphi^+(Q_2)}{\partial \vec{n}} - \epsilon_2 \frac{\partial \varphi^-(Q_2)}{\partial \vec{n}} = 0, \quad Q_2 \in S_2 \tag{12}$$

On the basis of the theorem of single layer potential normal derivative jump in the boundary surface points, we differentiate (10) in $Q_1 \in S_1$ and $Q_2 \in S_2$ and we obtain:

$$\frac{\partial \varphi^+(Q_1)}{\partial \vec{n}} = \frac{\sigma_1(Q_1)}{2\epsilon^e} - \frac{1}{4\pi\epsilon^e} \left\{ \oint_{S_1} \sigma_1(M_1) \frac{\cos\psi_{Q_1M_1}}{r_{Q_1M_1}^2} dS_{M_1} + \right.$$

$$\left. + \oint_{S_2} \sigma_2(M_2) \frac{\cos\psi_{Q_2M_2}}{r_{Q_2M_2}^2} dS_{M_2} \right\} -$$

$$-\frac{1}{4\pi\epsilon^e}\left\{\oint_{V_1}\rho_1'(N_1)\frac{\cos\psi_{Q_1N_1}}{r^2_{Q_1N_1}}dV_{N_1}+\right.$$

$$\left.-\oint_{V_e}\rho_e(N)\frac{\cos\psi_{Q_1N}}{r^2_{Q_1N}}dV_N\right\} \quad (13)$$

Similarly, by differentiating the potential (10) in the direction normal to the inside of surface S_1 we obtain:

$$\frac{\partial\varphi^-(Q_1)}{\partial\vec{n}}=-\frac{\sigma_1(Q_1)}{2\epsilon^e}-\frac{1}{4\pi\epsilon^e}\left\{\oint_{S_1}\sigma_1(M_1)\frac{\cos\psi_{Q_1M_1}}{r^2_{Q_1M_1}}dS_{M_1}+\right.$$

$$\left.+\oint_{S_2}\sigma_2(M_2)\frac{\cos\psi_{Q_2M_2}}{r^2_{Q_2M_2}}dS_{M_2}\right\}-$$

$$-\frac{1}{4\pi\epsilon^e}\left\{\oint_{V_1}\rho_1'(N_1)\frac{\cos\psi_{Q_1N_1}}{r^2_{Q_1N_1}}dV_{N_1}+\right.$$

$$\left.-\oint_{V_e}\rho_e(N)\frac{\cos\psi_{Q_1N}}{r^2_{Q_1N}}dV_N\right\} \quad (14)$$

It should be stressed that the integral formula (10) describes the potential distribution in the whole space $\mathbb{R}^3$ filled with dielectic ϵ^e as a result of the performed medium transformation. By determining this time the normal derivatives of potential (10) in points on surface $Q_2 \in S_2$ we obtain:

$$\frac{\partial\varphi^+(Q_2)}{\partial\vec{n}}=\frac{\sigma_2(Q_2)}{2\epsilon^e}-\frac{1}{4\pi\epsilon^e}\left\{\oint_{S_2}\sigma_1(M_2)\frac{\cos\psi_{Q_2M_2}}{r^2_{Q_2M_2}}dS_{M_2}+\right.$$

$$\left.+\oint_{S_2}\sigma_2(M_2)\frac{\cos\psi_{Q_2M_2}}{r^2_{Q_2M_2}}dS_{M_2}\right\}-$$

$$-\frac{1}{4\pi\epsilon^e}\left\{\oint_{V_2}\rho_2'(N_2)\frac{\cos\psi_{Q_2N_2}}{r^2_{Q_2N_2}}dV_{N_2}+\right.$$

$$\left.+\oint_{V_e}\rho_e(N)\frac{\cos\psi_{Q_2N}}{r^2_{Q_2N}}dV_N\right\} \quad (15)$$

Similarly by differentiating the potential (10) in the direction normal to the inside of surface S_2 we obtain:

$$\frac{\partial\varphi^-(Q_2)}{\partial\vec{n}}=-\frac{\sigma_2(Q_2)}{2\epsilon^e}-\frac{1}{4\pi\epsilon^e}\left\{\oint_{S_2}\sigma_2(M_2)\frac{\cos\psi_{Q_2M_2}}{r^2_{Q_2M_2}}dS_{M_2}+\right.$$

$$\left.+\oint_{S_2}\sigma_2(M_2)\frac{\cos\psi_{Q_2M_2}}{r^2_{Q_2M_2}}dS_{M_2}\right\}-$$

$$-\frac{1}{4\pi\epsilon^e}\left\{\oint_{V_2}\rho_2'(N_2)\frac{\cos\psi_{Q_2N_2}}{r^2_{Q_2N_2}}dV_{N_2}+\right.$$

$$\left.+\oint_{V_e}\rho_e(N)\frac{\cos\psi_{Q_2N}}{r^2_{Q_2N}}dV_N\right\} \quad (16)$$

After substituting the obtained direction derivatives to (11) and (12) and after performing appropriate transformations we obtain a system of two Fredholm integral equations of the second kind.

$$\sigma_1(Q_1)-\lambda_1\oint_{S_1}\mathcal{K}_{11}(Q_1M_1)\sigma_1(M_1)dS_{M_1}-$$

$$-\lambda_1\oint_{S_2}\mathcal{K}_{12}(Q_1M_2)\sigma_2(M_2)dS_{M_2}=f_1(Q_1) \quad (17)$$

$$\sigma_2(Q_2)-\lambda_2\oint_{S_1}\mathcal{K}_{21}(Q_2M_1)\sigma_1(M_1)dS_{M_1}-$$

$$-\lambda_2\oint_{S_2}\mathcal{K}_{22}(Q_2M_2)\sigma_2(M_2)dS_{M_2}=f_2(Q_2) \quad (18)$$

where $\mathcal{K}_{ij}(Q_iM_j)$, i,j=1,2 is the nucleus of the Fredholm's integral operator.

$$\mathcal{K}_{11}(Q_1M_1)=\oint_{S_1}\frac{\cos\psi_{Q_1M_1}}{2\pi\, r^2_{Q_1M_1}}dS_{M_1} \quad (19)$$

$$\mathcal{K}_{12}(Q_1M_2)=\oint_{S_2}\frac{\cos\psi_{Q_1M_2}}{2\pi\, r^2_{Q_1M_2}}dS_{M_2} \quad (20)$$

$$\mathcal{K}_{21}(Q_2M_1)=\oint_{S_1}\frac{\cos\psi_{Q_2M_1}}{2\pi\, r^2_{Q_2M_1}}dS_{M_1} \quad (21)$$

$$\mathcal{K}_{22}(Q_2M_2)=\oint_{S_2}\frac{\cos\psi_{Q_2M_2}}{2\pi\, r^2_{Q_2M_2}}dS_{M_2} \quad (22)$$

The parameter λ_k, k=1,2 is expressed by the dielectrics' permeability:

$$\lambda_1=\frac{\epsilon_1-\epsilon_e}{\epsilon_1+\epsilon_e},\quad \lambda_2=\frac{\epsilon_2-\epsilon_e}{\epsilon_2+\epsilon_e} \quad (23)$$

The right sides of (17), (18) are the source functions expressed by the original volume load densities.

$$f_1(Q_1)=\lambda_1\int_{V_1}\rho'(N_1)\frac{\cos\psi_{Q_1N_1}}{r^2_{Q_1N_1}}dV_{N_1}+$$

$$+\lambda_1\oint_{V_e}\rho_e(N)\frac{\cos\psi_{Q_1N}}{r^2_{Q_1N}}dV_N \quad (24)$$

$$f_2(Q_2)=\lambda_2\int_{V_1}\rho'(N_1)\frac{\cos\psi_{Q_2N_1}}{r^2_{Q_2N_1}}dV_{N_1}+$$

$$+ \lambda_2 \oint_{V_e} \rho_e(N) \frac{\cos \psi_{Q_2 N}}{r^2_{Q_2 N}} dV_N \quad (25)$$

The educed system (17), (18) is a Fredholm system of the second kind with a weakly singular nucleus. It has to be noted that for numerical reasons its spectral characteristics are especially important. The integral equations theory implies that the value of parameter $\lambda = 1$ is the eigenvalue of the Fredholm's nucleus. In our case it happens when the permeabilities ϵ_1 and ϵ_2 are many times higher than the permeability ϵ_e. If then ($\epsilon_1 \longrightarrow \infty$) and ($\epsilon_2 \rightarrow \infty$), we have:

$$\lambda_1 = \lim_{\epsilon_1 \rightarrow \infty} \left\{ \frac{\epsilon_1 - \epsilon_e}{\epsilon_1 + \epsilon_e} \right\} = 1; \quad \lambda_2 = \lim_{\epsilon_2 \rightarrow \infty} \left\{ \frac{\epsilon_2 - \epsilon_e}{\epsilon_2 + \epsilon_e} \right\} = 1 \quad (26)$$

λ_1 and λ_2 receive the eigenvalues of nucleuses from (17), (18). This causes problems associated with stability and ambiguity of the system's numerical solution. Such a case necessitates regularization of the nucleuses in order to shift the eigenvalue spectrum.

III. Generalized formulation of the integral problem

We shall formulate the system (17), (18) in a generalized form in the Galerkin sense. This leads to a system of two variational identities and a a system of two energy potentials in the Wainberg sense.

A. *Galerkin's variational identities*

In a normalized space of functions integrable in square, equipped with a scalar product (u, v) and a norm $\| u \|$ where $u, v \in L2(S_{S_k})$, we shall introduce a bi-linear form $a(\sigma, v) : L^2(S_k) \times L^2(S_k) \longrightarrow R^1$ and a linear form $l(v) : L^2(S_k) \longrightarrow R^1$, where $v(x) \in L^2(S_k)$ is a test function. For the system (17), (18) these forms have the form of surface integrals.

$$< \sigma_1, v_1 >= \oint_{S_1} \sigma_1(Q_1) v_1(Q_1) dS_{Q_1} \quad (27)$$

$$< \sigma_2, v_2 >= \oint_{S_2} \sigma_2(Q_2) v_2(Q_2) dS_{Q_2} \quad (28)$$

$$l_1(v_1) =< f_1, v_1 >= \oint_{S_1} f_1(Q_1) v_1(Q_1) dS_{Q_1} \quad (29)$$

$$l_2(v_2) =< f_2, v_2 >= \oint_{S_1} f_1(Q_1) v_1(Q_1) dS_{Q_1} \quad (30)$$

In accordance, the bi-linear forms $a_{ij}(\sigma_i, v_j)$ i,j=1,2 are expressed with double integrals.

$$a_{11}\ (\sigma_1, v_1) =< \lambda_1\ \mathcal{K}_{11}(\sigma_1)\ ;,\ v_1 >=$$

$$= \oint_{S_1} \lambda_1 \oint_{S_1} \sigma_1(M_1) \frac{\cos \psi_{Q_1 M_1}}{r^2_{Q_1 M_1}} dS_{M_1} v_1(Q_1) dS_{Q_1} \quad (31)$$

$$a_{12}(\sigma_2, v_1) =< \lambda_1\ \mathcal{K}_{12}(\sigma_2)\ ;,\ v_1 >=$$

$$= \oint_{S_1} \lambda_1 \oint_{S_2} \sigma_2(M_2) \frac{\cos \psi_{Q_1 M_2}}{r^2_{Q_1 M_2}} dS_{M_2} v_1(Q_1) dS_{Q_1} \quad (32)$$

$$a_{21}(\sigma_1, v_2) =< \lambda_2\ \mathcal{K}_{21}(\sigma_1)\ ;,\ v_2 >=$$

$$= \oint_{S_2} \lambda_2 \oint_{S_1} \sigma_1(M_1) \frac{\cos \psi_{Q_2 M_1}}{r^2_{Q_2 M_1}} dS_{M_1} v_2(Q_1) dS_{Q_2} \quad (33)$$

$$a_{22}(\sigma_2, v_2) =< \lambda_2\ \mathcal{K}_{22}(\sigma_2)\ ;,\ v_2 >=$$

$$= \oint_{S_2} \lambda_2 \oint_{S_2} \sigma_2(M_2) \frac{\cos \psi_{Q_2 M_2}}{r^2_{Q_2 M_2}} dS_{M_2} v_2(Q_2) dS_{Q_2} \quad (34)$$

The system expressed by the integral forms (27)-(34) receives the form of two variational identities:

$$\begin{array}{l} < \sigma_1, v_1 > -a_{11}(\sigma_1, v_1) - a_{12}(\sigma_2, v_1) = l(v_1) \\ < \sigma_2, v_2 > -a_{21}(\sigma_1, v_2) - a_{22}(\sigma_2, v_2) = l(v_2) \end{array} \quad (35)$$

For the matrix representation of the system of identities (35) we shall now give a theorem of solution existence and unambiguouty. By introducing vectors

$$\sigma = \left[\begin{array}{c} \sigma_1 \\ \sigma_2 \end{array} \right], \quad \vec{v} = \left[\begin{array}{c} v_1 \\ v_2 \end{array} \right], \quad l(\vec{v}) = \left[\begin{array}{c} l_1(v_1) \\ l_2(v_2) \end{array} \right] \quad (36)$$

$$\vec{a}(\sigma, \vec{v}) = \left[\begin{array}{c} (\sigma_1, v_1) - a_{11}(\sigma_1, v_1) - a_{12}(\sigma_2, v_1) \\ (\sigma_2, v_2) - a_{21}(\sigma_1, v_2) - a_{22}(\sigma_2, v_2) \end{array} \right] \quad (37)$$

we obtain a solution

$$\vec{a}(\sigma, \vec{v}) = \vec{l}(\vec{v}), \quad \bigwedge \vec{l}(\vec{v}) \in \mathcal{V}'(S_k) \quad (38)$$

For the variational identity (38) we will formulate a multi-surface boundary problem.
(Question 2): For the given linear form $\vec{l}(\vec{v}) \in \mathcal{V}'(S_k)$ and any test vector $\vec{v} \in L^2(S_k)$ determine the secondary sources vector $\sigma \in L^2(S_k)$ that fulfills the variational identity (38). The bi-linear form $\vec{a}(\sigma, \vec{v})$ and linear form $\vec{l}(\vec{v})$ fulfill necessary and satisfactory conditions for the existence and unambiguouty of the equation's solution, the variational identity in the Galerkin sense (38).

B. *Wainberg's potentials*

Now we shall write the integral equation system (17)-(18) in the operator form.

$$\mathcal{P}(\sigma) = f \quad (39)$$

where the operator $\mathcal{P}$ and the righ side f have the form:

$$\mathcal{P} = \vec{I} - \lambda\ \mathcal{K}(Q, M) \quad (40)$$

$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{41}$$

The identity operator $\vec{I}$ and λ are a second-order matrices,

$$\vec{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{42}$$

And $\mathcal{K}(\mathcal{Q}, \mathcal{M})$ is the Fredholm nucleus operator matrix $\mathcal{K}_{ij}(Q_i, M_j)$, i,j=1,2 expressed by (19)-(22).

$$\mathcal{K}(\mathcal{Q}, \mathcal{M}) = \begin{bmatrix} \mathcal{K}_{11}(Q_1 M_1) & \mathcal{K}_{12}(Q_1 M_2) \\ \mathcal{K}_{21}(Q_1 M_2) & \mathcal{K}_{22}(Q_2 M_2) \end{bmatrix} \tag{43}$$

The operator equation (38) obtains the following form in the matrix representation.

$$\left\{ \vec{I} - \lambda\, \mathcal{K}(Q, M) \right\} \sigma(Q) = f(Q) \tag{44}$$

By using the potentiality operators theory [7] we shall formulate the integral equation system (17), (18) as two electostatic energy field functionals, of the energy stored on the boundary surfaces S_k expressed by the surface densities σ_k and the original sources energy with density ρ_k. We shall use the following theorem:

(Theorem 1): If the operator $P : (L^2(S_k) \times L^2(S_k)) \to L^2(S_k)$ is differentiable in the Gateaux derivative sense in every point of $\sigma \in L^2(S_k)$, i.e. it has a derivative $d\mathcal{P}(\sigma; u)$ in point σ in direction u, where,

$$d\mathcal{P}(\sigma; u) = \lim_{t \to 0} \frac{1}{t} \{\mathcal{P}(\sigma + tu) - \mathcal{P}(\sigma)\} =$$

$$= \frac{d}{dt} \mathcal{P}(\sigma + tu) \mid_{t=0} \tag{45}$$

and the functional $< \mathcal{P}(\sigma; u), v >$ is continuous in point $\sigma \in L^2(S_k)$ then the necessary and satisfactory condition for the operator $\mathcal{P}$ to be a potential operator is:

$$< \mathcal{P}(\sigma; u), v > = < \mathcal{P}(\sigma; v), u > \tag{46}$$

This means that the bi-linear functional $< \mathcal{P}(\sigma; u), v >$ in points u and v has to be symmetrical for every density $\sigma \in L^2(S_k)$.

(Theorem 2): If the operator $\mathcal{P} : (L^2(S_k) \times L^2(S_k)) \to L^2(S_k)$ is continuous and potential then there exists a functional $\mathcal{F}(\sigma)$ such that its gradient is equal to this operator in this point.

$$\mathcal{P}(\sigma) = grad\, \mathcal{F}(\sigma) = \nabla \otimes \mathcal{F}(\sigma) \tag{47}$$

This function is given by:

$$\mathcal{F}(\sigma) = \int_o^1 < \mathcal{P}(t\sigma) - f, \sigma > dt \tag{48}$$

We shall verify that the operator (40) fulfills the necessary and satisfactory condition for potentiality. We shall determine the direction derivative (45) and we will check the symmetry condition (46).

$$d\mathcal{P}(\sigma; \vec{u}) = \frac{d}{dt} \mathcal{P}(\sigma + tu) \mid_{t=0} = \vec{u} - \lambda \mathcal{K}(Q, M) \vec{u} \tag{49}$$

$$d\mathcal{P}(\sigma; \vec{v}) = \frac{d}{dt} \mathcal{P}(\sigma + tv) \mid_{t=0} = \vec{v} - \lambda \mathcal{K}(Q, M) \vec{v} \tag{50}$$

$$< \vec{u} - \lambda \mathcal{K}(Q, M)\vec{u}, \vec{v} > = < \vec{v} - \lambda \mathcal{K}(Q, M)\vec{v}, \vec{u} > \tag{51}$$

It is easy to state that the the symmetry condition is fulfilled and thus the operator $\mathcal{P}$ is potential. The potential (48) obtains the following form through integration:

$$\mathcal{F}(\sigma) = \int_o^1 < \mathcal{P}(t\,\sigma) - \vec{f}, \sigma > dt =$$

$$= \int_o^1 < t\, \mathcal{P}(\sigma) - \vec{f}, \sigma > dt =$$

$$= \int_o^1 < t\, \mathcal{P}(\sigma), \sigma > dt - \int_o^1 < \vec{f}, \sigma > dt =$$

$$= \frac{1}{2} < \mathcal{P}(\sigma), \sigma > - < \vec{f}, \sigma > \tag{52}$$

After substituting the vectors (36), (41) and the operator (40) to (52) we obtain a direct form of the potential $\mathcal{F}(\sigma)$ whose elements are expressed by appropriate scalar products in the space $L^2(S_k) \times L^2(S_k)$, $< \cdot, \cdot >$.

$$\mathcal{F}(\sigma) = \frac{1}{2}(< \sigma_1, v_1 > + < \sigma_2, v_2 >) -$$

$$-\frac{1}{2}(< \lambda_1\, \mathcal{K}_{11}(\sigma_1)\ ,\ v_1 > + < \lambda_1\, \mathcal{K}_{12}(\sigma_2)\ ,\ v_1 > +$$

$$+ < \lambda_1\, \mathcal{K}_{12}(\sigma_2)\ ,\ v_1 > + < \lambda_1\, \mathcal{K}_{12}(\sigma_{12})\ ,\ v_2 >) -$$

$$- < f_1\ ,\ v_1 > - < f_2\ ,\ v_2 > \tag{53}$$

And in the integral form:

$$\mathcal{F}(\sigma) = \frac{1}{2} \left\{ \oint_{S_1} \sigma_1(Q_1) v_1(Q_1) dS_{Q_1} + \right.$$

$$\left. + \oint_{S_2} \sigma_2(Q_2) v_2(Q_2) dS_{Q_2} \right\} -$$

$$-\frac{1}{2} \left\{ \oint_{S_1} \lambda_1 \oint_{S_1} \sigma_1(M_1) \frac{\cos \psi_{Q_1 M_1}}{r^2_{Q_1 M_1}} dS_{M_1} v_1(Q_1) dS_{Q_1} + \right.$$

$$+\oint_{S_1}\lambda_1\oint_{S_2}\sigma_2(M_2)\frac{\cos\psi_{Q_1M_2}}{r^2_{Q_1M_2}}dS_{M_2}v_1(Q_1)dS_{Q_1}+$$

$$+\oint_{S_2}\lambda_2\oint_{S_1}\sigma_1(M_1)\frac{\cos\psi_{Q_2M_1}}{r^2_{Q_2M_1}}dS_{M_1}v_2(Q_1)dS_{Q_2}+$$

$$+\oint_{S_2}\lambda_2\oint_{S_1}\sigma_1(M_1)\frac{\cos\psi_{Q_2M_1}}{r^2_{Q_2M_1}}dS_{M_1}v_2(Q_1)dS_{Q_2}\Bigg\}-$$

$$-\oint_{S_1}f_1(Q_1)v_1(Q_1)dS_{Q_1}-\oint_{S_2}f_2(Q_2)v_2(Q_2)dS_{Q_2} \quad (54)$$

The original source functions $f_1(Q_1)$ i $f_2(Q_2)$ are defined by (24), (25). We shall formulate a variational problem for the functional (53) of obtaining the extreme of ther functional $\mathcal{F}$.

(Question 3): In the set of acceptable density distributions in the single layer $v \in \mathcal{U}_{ad}(S_k) \subset L^2(S_k) \times L^2(S_k)$ determine the density σ which gives the lower boundary of the functional (53).

$$\mathcal{F}(\sigma) = \inf_{\vec{v}\in U_{ad}} \mathcal{F}(\vec{v}) \quad (55)$$

If the functional (53) is weakly pulled from the bottom and is also radially non-convergent then it reaches a global minimum in the definitive space. It can also be shown that the point $(\sigma = (\sigma_1, \sigma_2))$ that gives the global minimum to the functional $\mathcal{F}(\sigma)$ is a solution of (17), (18). Also this solution is equivalent to the solution of the variational identities in the Galerkin sense (Question 2).

IV. Final Remarks

The proposed method of integral equations based on the single and double layer potentials is a universal method of formulating boundary algorithms for non-homogeneous, anizotropic and non-linear systems. It allows for mathematical modeling leading to the boundary element method BEM for systems with unlimited boundary surfaces. In case when the material medium is non-homogeneous with high disproportions of permeability values, the equation system needs regularization which shifts the spectrum of the system so that the λ_k parameter is not equal to one.

The case where the λ parameter is equal or close to one, is associated with ambiguity of the integral system solution and with numerical instability of algorithms based on the boundary finite element approximation.

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