Realization Problem for Positive Multivariable Linear Systems with Time-Delay

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***Abstract* — The realization problem for positive multivariable discrete-time systems with one time-delay is formulated and solved. Conditions for the solvability of the realization problem are established. A procedure for computation of a minimal positive realization of a proper rational matrix is presented and illustrated by an example**

I. INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [4, 5]. Recent developments in positive systems theory and some new results are given in [6]. Realizations problem of positive linear systems without time delays has been considered in many papers and books [1,4, 5]. Explicit solution of equations describing the discrete-time systems with time-delay has been given in [2]. Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [3, 7].

In this paper the realization problem for positive multivariable discrete-time systems with time-delay will be formulated and solved. Conditions for the solvability of the realization problem will be established and a procedure for computation of a minimal positive realization of a proper rational matrix will be presented.

To the best knowledge of the author the realization problem for positive linear systems with time-delays has not been considered yet.

Consider the multivariable discrete-time linear system with one time-delay

II. PROBLEM FORMULATION

$$x_{i+1} = A_0 x_i + A_1 x_{i-1} + Bu_i \quad i \in Z_+ = \{0,1,\ldots\} \tag{1a}$$

$$y_i = Cx_i + Du_i \tag{1b}$$

where $x_i \in R^n$, $u_i \in R^m$, $y_i \in R^p$ are the state vector, input and output, respectively and $A_k \in R^{n\times n}$, $k = 0,1$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$.

Initial conditions for (1a) are given by

$$x_{-1}, x_0 \in R^n \tag{2}$$

Let $R_+^{n\times m}$ be the set of $n\times m$ real matrices with nonnegative entries and $R_+^n = R_+^{n\times 1}$.

Definition 1 [3]. The system (1) is called (internally) positive if for every $x_{-1}, x_0 \in R_+^n$ and all inputs $u_i \in R_+^m$, $i \in Z_+$ we have $x_i \in R_+^n$ and $y_i \in R_+^p$ for $i \in Z_+$.

Theorem 1 [3]. The system (1) is positive if and only if

$$A_0 \in R_+^{n\times n}, A_1 \in R_+^{n\times n}, B \in R_+^{n\times m}, C \in R_+^{p\times n}, D \in R_+^{p\times m} \tag{3}$$

The transfer matrix of (1) is given by

$$T(z) = C\left[I_n z - A_0 - A_1 z^{-1}\right]^{-1} B + D \tag{4}$$

Definition 2. Matrices (3) are called a positive realization of a given proper rational matrix $T(z)$ if and only if they satisfy the equality (4). A realization (3) is called minimal if and only if the dimension n of A_0 and A_1 is minimal among all realizations of $T(z)$.

The positive realization problem can be stated as follows. Given a proper rational matrix $T(z)$. Find a positive realization (3) of the rational matrix $T(z)$.

Conditions for the solvability of the problem will be established and a procedure for computation of a positive realization will be presented.

III. PROBLEM SOLUTION

The transfer matrix (4) can be rewritten in the form

$$T(z) = C\left[z^{-1}\left(I_n z^2 - A_0 z - A_1\right)\right]^{-1} B + D = \\ = \frac{Cz \operatorname{Adj}[I_n z^2 - A_0 z - A_1]B}{\det\left[I_n z^2 - A_0 z - A_1\right]} + D = \frac{zN(z)}{d(z)} + D \tag{5}$$

where

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$$\begin{aligned} N(z) &= C \operatorname{Adj}\left[I_n z^2 - A_0 z - A_1\right]B = \\ &= N_{2(n-1)} z^{2(n-1)} + N_{2n-3} z^{2n-3} + \ldots + N_1 z + N_0 \\ d(z) &= \det\left[I_n z^2 - A_0 z - A_1\right] = \\ &= z^{2n} - a_{2n-1} z^{2n-1} - \ldots - a_1 z - a_0 \end{aligned} \tag{6}$$

and $\operatorname{Adj}\left[I_n z^2 - A_0 z - A_1\right]$ denotes the adjoint matrix for $\left[I_n z^2 - A_0 z - A_1\right]$

From (5) we have

$$D = \lim_{z\to\infty} T(z) \tag{7}$$

since $\lim_{z\to\infty} \left[z^{-1}\left(I_n z^2 - A_0 z - A_1\right)\right]^{-1} = 0$.

The strictly proper part of $T(z)$ is given by

$$T_{sp}(z) = T(z) - D = \frac{zN(z)}{d(z)} \tag{8}$$

Therefore, the positive realization problem has been reduced to finding matrices

$$A_0 \in R_+^{n\times n}, A_1 \in R_+^{n\times n}, B \in R_+^{n\times m}, C \in R_+^{p\times n} \tag{9}$$

for a given strictly proper rational matrix (8).

Lemma 1. The strictly proper transfer matrix (8) has the form

$$T'_{sp}(z) = \frac{N(z)}{d'(z)} \tag{10}$$

if and only if $\det A_1 = 0$ where

$$d'(z) = z^{2n-1} - a_{2n-1} z^{2n-2} - \quad - a_2 z - a_1 \tag{11}$$

Proof. From definition (6) of $d(z)$ for $z = 0$ it follows that $a_0 = \det A_1$. Note that $d(z) = zd'(z)$ if and only if $a_0 = 0$ and (8) can be reduced to (10). ■

Lemma 2. If the matrices A_0 and A_1 have one of the following forms

$$A_0 = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ a_1 & 0 & \ldots & 0 & 0 \\ a_3 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-7} & 0 & \ldots & 0 & 0 \\ a_{2n-5} & 0 & \ldots & 0 & a_{2n-3} \\ 0 & 0 & \ldots & 0 & a_{2n-1} \end{bmatrix} \in R^{n\times n},$$

$$A_1 = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ a_0 & 0 & \ldots & 0 & 0 & 0 \\ a_2 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{2(n-4)} & 0 & \ldots & 0 & 0 & 0 \\ a_{2(n-3)} & 0 & \ldots & 1 & 0 & a_{2(n-2)} \\ 0 & 0 & \ldots & 0 & 1 & a_{2(n-1)} \end{bmatrix} \in R^{n\times n} \tag{12a}$$

$$\overline{A}_0 = \begin{bmatrix} 0 & a_1 & a_3 & \ldots & a_{2n-7} & a_{2n-5} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & a_{2n-3} & a_{2n-1} \end{bmatrix},$$

$$\overline{A}_1 = \begin{bmatrix} 0 & a_0 & a_2 & \ldots & a_{2(n-4)} & a_{2(n-3)} & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & a_{2(n-2)} & a_{2(n-1)} \end{bmatrix} \tag{12b}$$

$$\hat{A}_0 = \begin{bmatrix} a_{2n-1} & 0 & \ldots & 0 & 0 \\ a_{2n-3} & 0 & \ldots & 0 & a_{2n-5} \\ 0 & 0 & \ldots & 0 & a_{2n-7} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & a_3 \\ 0 & 0 & \ldots & 0 & a_1 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

$$\hat{A}_1 = \begin{bmatrix} a_{2(n-1)} & 1 & 0 & \ldots & 0 & 0 \\ a_{2(n-2)} & 0 & 1 & \ldots & 0 & a_{2(n-3)} \\ 0 & 0 & 0 & \ldots & 0 & a_{2(n-4)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & a_2 \\ 0 & 0 & 0 & \ldots & 0 & a_0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \tag{12c}$$

$$\widetilde{A}_0 = \begin{bmatrix} a_{2n-1} & a_{2n-3} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_{2n-5} & a_{2n-7} & \dots & a_3 & a_1 & 0 \end{bmatrix},$$

$$\widetilde{A}_1 = \begin{bmatrix} a_{2(n-1)} & a_{2(n-2)} & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & a_{2(n-3)} & a_{2(n-4)} & \dots & a_2 & a_0 & 0 \end{bmatrix} \quad (12d)$$

then

$$\det\left[I_n z^2 - A_0 z - A_1\right] = \det\left[I_n z^2 - \overline{A}_0 z - \overline{A}_1\right] = $$
$$= \det\left[I_n z^2 - \hat{A}_0 z - \hat{A}_1\right] = \det\left[I_n z^2 - \widetilde{A}_0 z - \widetilde{A}_1\right] =$$
$$= z^{2n} - a_{2n-1} z^{2n-1} - a_{2(n-1)} z^{2(n-1)} - \dots - a_1 z - a_0 \quad (13)$$

Proof. Expansion of the determinant with respect to the first row yields

$$\det\left[I_n z^2 - A_0 z - A_1\right] =$$
$$= \begin{vmatrix} z^2 & 0 & \dots & 0 & 0 & -1 \\ -a_1 z - a_0 & z^2 & \dots & 0 & 0 & 0 \\ -a_3 z - a_2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -a_{2n-7} z - a_{2(n-4)} & 0 & \dots & z^2 & 0 & 0 \\ -a_{2n-5} z - a_{2(n-3)} & 0 & \dots & -1 & z^2 & -a_{2n-3} z - a_{2(n-2)} \\ 0 & 0 & \dots & 0 & -1 & z^2 - a_{2n-1} z - a_{2(n-1)} \end{vmatrix} =$$
$$z^{2(n-2)}\left(z^4 - a_{2n-1} z^3 - a_{2(n-1)} z^2 - a_{2n-3} z - a_{2(n-2)}\right) + (-1)^{n+2}$$
$$\times \begin{vmatrix} -a_1 z - a_0 & z^2 & 0 & \dots & 0 & 0 \\ -a_3 z - a_2 & -1 & z^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{2n-7} z - a_{2(n-4)} & 0 & 0 & \dots & z^2 & 0 \\ -a_{2n-5} z - a_{2(n-3)} & 0 & 0 & \dots & -1 & z^2 \\ 0 & 0 & 0 & \dots & 0 & -1 \end{vmatrix} = \dots =$$
$$= z^{2n} - a_{2n-1} z^{2n-1} - a_{2(n-1)} z^{2(n-1)} - \dots - a_1 z - a_0 \quad (13)$$

The proof for (12b) follows from the fact that $\overline{A}_0 = A_0^T, \overline{A}_1 = A_1^T$ and

$$\det\left[I_n z^2 - \overline{A}_0 z - \overline{A}_1\right] = \det\left[I_n z^2 - A_0 z - A_1\right]^T$$

where T stands for transpose.

It is easy to verify that $\hat{A}_0 = P A_0 P$, $\hat{A}_1 = P A_1 P$ where

$$P = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Taking into account that $P^{-1} = P^T = P$ we obtain

$$\det\left[I_n z^2 - \hat{A}_0 z - \hat{A}_1\right] = \det\left[I_n z^2 - A_0 z - A_1\right]$$

Finally note that $\widetilde{A}_0 = \hat{A}_0^T$ and $\widetilde{A}_1 = \hat{A}_1^T$. ■

The matrices A_0 and A_1 having one of the forms (12) will be called the matrices in canonical forms.
The following two remarks are in order.

Remark 1. The matrices (12) have nonnegative entries if and only if the coefficients $a_k, k = 0,1,\dots,2n-1$ of the polynomial (13) are nonnegative.

Remark 2. The dimension $n \times n$ of matrices (12) is the smallest possible one for (8).

Definition 3. A pair (A_0, A_1) of square matrices $A_0, A_1 \in R^{n \times n}$ is called cyclic if and only if its characteristic polynomial

$$d(z) = \det\left[I_n z^2 - A_0 z - A_1\right] = $$
$$= z^{2n} - a_{2n-1} z^{2n-1} - \dots - a_1 z - a_0 \quad (14)$$

is equal to the minimal polynomial $\Psi(z)$ of the pair, i.e. $d(z) = \Psi(z)$.
It is well-known that the polynomials are related by

$$\Psi(z) = \frac{d(z)}{D_{n-1}(z)} \quad (15)$$

and that

$$\Psi(z) = d(z) \quad (16)$$

if and only if $D_{n-1}(z) = 1$ or equivalently

$$i_1(z) = i_2(z) = \dots = i_{n-1}(z) \quad (17)$$

where $D_{n-1}(z)$ is the greatest common divisor of all $n-1$ order minors of the matrix $\left[I_n z^2 - A_0 z - A_1\right]$ and $i_k(z), k = 1,\dots n-1$ are the its monic invariant polynomials.

Lemma 3. Every pair of the matrices (12) is cyclic for any values of its parameters a_k , $k = 1,2,...2n-1$.

Proof. The details of the proof will be given only for the pair (12a). In the remaining cases the proof is similar.
Note that the $n-1$ order minor obtained by removing the second row and the first column of the matrix

$$\left[I_n z^2 - A_0 z - A_1\right] = \begin{bmatrix} z^2 & 0 & \dots & 0 & 0 & -1 \\ -a_1 z - a_0 & z^2 & \dots & 0 & 0 & 0 \\ -a_3 z - a_2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -a_{2n-7} z - a_{2(n-4)} & 0 & \cdots & z^2 & 0 & 0 \\ -a_{2n-5} z - a_{2(n-3)} & 0 & \dots & -1 & z^2 & -a_{2n-3} z - a_{2(n-3)} \\ 0 & 0 & \dots & 0 & -1 & z^2 - a_{2n-1} z - a_{2(n-1)} \end{bmatrix} \tag{18}$$

is equal to $(-1)^{n-1}$. Therefore $D_{n-1}(z) = 1$ and by (15) $\Psi(z) = d(z)$. ■

For any square matrices $A_0, A_1 \in R^{n\times n}$ the inverse matrix $\left[I_n z^2 - A_0 z - A_1\right]^{-1}$ can be written in the form

$$\left[I_n z^2 - A_0 z - A_1\right]^{-1} = \frac{\overline{N}(z)}{d(z)} \tag{19}$$

where $\overline{N}(z)$ is an $n\times n$ polynomial matrix and $d(z)$ is a polynomial. The matrix (19) is called in the standard form if the matrix $\frac{\overline{N}(z)}{d(z)}$ is irreducible and the leading coefficient of $d(z)$ is equal to 1.

Definition 4. The matrix (19) is called normal if and only if, every nonzero second order minor of the polynomial matrix $\overline{N}(z)$ is divisible (with zero remainder) by $d(z)$.

Lemma 4. The standard matrix (19) for $n \geq 2$ is normal if and only if the pair (A_0, A_1) is cyclic.

Proof. Let the pair (A_0, A_1) be cyclic. Then by definition 3, (16) and (17) hold and the Smith canonical form of $\left[I_n z^2 - A_0 z - A_1\right]$ is equal to

$$\left[I_n z^2 - A_0 z - A_1\right]_S = \operatorname{diag}\left[1 \quad 1 \quad \dots \quad 1 \quad d(z)\right]$$
(20)
The adjoint matrix to (20) is given by

$$\begin{aligned} &\operatorname{Adj}\left[I_n z^2 - A_0 z - A_1\right]_S = \\ &= \operatorname{diag}\left[d(z) \quad d(z) \quad \dots \quad d(z) \quad 1\right] \end{aligned} \tag{21}$$

and every nonzero second order minor of (21) is divisible by $d(z)$. By Binet-Cauchy theorem every nonzero second order minor of the matrix $V(z)\operatorname{Adj}\left[I_n z^2 - A_0 z - A_1\right]_S U(z)$ is also divisible by $d(z)$ since it is the sum of products of second order minors of the unimodular matrices $V(z)$, $U(z)$ and of (21). The necessity will be shown by contradiction. By assumption the matrix (19) is irreducible. If the characteristic polynomial (14) is not equal to the minimal one $\Psi(z)$, $\Psi(z) \neq d(z)$ then by (15) $D_{n-1}(z) \neq 1$ and every nonzero $n-1$ order minor of $\left[I_n z^2 - A_0 z - A_1\right]$ is divisible by $D_{n-1}(z)\det\left[I_n z^2 - A_0 z - A_1\right] = D_{n-1}(z)\overline{d}(z)$ and the matrix (19) is reducible. So we get a contradiction. ■

Lemma 5. If the pair (A_0, A_1) has the canonical form (12a) then the adjoint matrix $\operatorname{Adj}\left[I_n z^2 - A_0 z - A_1\right]$ can be decomposed as follows

$$\begin{aligned} &\operatorname{Adj}\left[I_n z^2 - A_0 z - A_1\right] = \\ &= \overline{P}(z)\overline{Q}(z) + d(z)\overline{G}(z) \end{aligned} \tag{22a}$$

where

$$\overline{P}(z) = \begin{bmatrix} 1 \\ z^{2(n-1)} - a_{2n-1} z^{2n-3} - \dots - a_3 z - a_2 \\ z^{2(n-2)} - a_{2n-1} z^{2n-5} - \dots - a_5 z - a_4 \\ \vdots \\ z^4 - a_{2n-1} z^3 - a_{2(n-2)} z^2 \\ z^2 \end{bmatrix} \tag{22b}$$

$$\overline{Q}(z) = \left[z^{2(n-1)} - a_{2n-1} z^{2n-3} - \dots - a_{2(n-2)} z^{2(n-3)} \quad 1 \quad z^2 \quad z^4 \quad \dots \quad z^{2(n-2)}\right]$$

$$\overline{G}(z) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ * & 0 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & \dots & * & * \\ * & 0 & \dots & * & * \end{bmatrix}$$

$*$ denotes the entries that are not important in the considerations.
Similar decompositions hold for the pairs (12b), (12c) and (12d).

Proof. The adjoint matrix has the form

$$\mathrm{Adj}\left[I_n z^2 - A_0 z - A_1\right] =$$

$$\begin{bmatrix} z^{2(n-1)} - a_{2n-1} z^{2n-3} - \ldots - a_{2(n-2)} z^{2(n-3)} \\ * \\ * \\ \vdots \\ * \\ * \end{bmatrix}$$

$$\begin{bmatrix} 1 & z^2 & \cdots & z^{2(n-2)} \\ z^{2(n-1)} - a_{2n-1} z^{2n-3} - \ldots - a_3 z - a_2 & * & \cdots & * \\ z^{2(n-2)} - a_{2n-1} z^{2n-5} - \ldots - a_5 z - a_4 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ z^4 - a_{2n-1} z^3 - a_{2(n-2)} z^2 & * & \cdots & * \\ z^2 & * & \cdots & * \end{bmatrix} \quad (23)$$

and it can be written in the form (22) since by lemma 4 every nonzero second order minor of (23) is divisible by $d(z)$. It is easy to verify that (22b) satisfies (22a). ∎

Substitution of (22a) into (8) yields

$$T_{sp} = \frac{Cz\,\mathrm{Adj}\left[I_n z^2 - A_0 z - A_1\right]B}{\det\left[I_n z^2 - A_0 z - A_1\right]} = \frac{zP_c(z)Q_b(z)}{d(z)} + Cz\overline{G}(z)B \quad (24)$$

where

$$P_c(z) = C\overline{P}(z), \quad Q_b(z) = \overline{Q}(z)B \quad (25)$$

Remark 3. From (24) it follows that the positive realization (9) of (8) is independent of the polynomial matrix $\overline{G}(z)$ ($Cz\overline{G}(z)B$).

Using (22b) and (25) we obtain

$$P_c(z) = C\overline{P}(z) = \begin{bmatrix} C_1 & C_2 & \ldots & C_n \end{bmatrix} \times \begin{bmatrix} 1 \\ z^{2(n-1)} - a_{2n-1} z^{2n-3} - \ldots - a_3 z - a_2 \\ z^{2(n-2)} - a_{2n-1} z^{2n-5} - \ldots - a_5 z - a_4 \\ \vdots \\ z^4 - a_{2n-1} z^3 - a_{2(n-2)} z^2 \\ z^2 \end{bmatrix} = \quad (26a)$$

$$\begin{aligned} &= C_2 z^{2(n-1)} - C_2 a_{2n-1} z^{2n-3} + \\ &+ \left(C_3 - a_{2(n-1)} C_2\right) z^{2(n-2)} - \ldots + \\ &+ \left(C_n - a_{2(n-1)} C_{n-1} - \ldots - a_6 C_3 - a_4 C_2\right) z^2 + \ldots \\ &- \left(\ldots - a_5 C_3 - a_3 C_2\right) z + \\ &+ C_1 - a_2 C_2 - a_4 C_3 \ldots \end{aligned}$$

$$\begin{aligned} Q_b(z) &= \overline{Q}(z)B = \\ &= \left[z^{2(n-1)} - a_{2n-1} z^{2n-3} - \ldots - a_{2(n-2)} z^{2(n-3)}\right. \\ &\left. 1 \;\; z^2 \;\; z^4 \cdots z^{2(n-2)}\right] \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} = \\ &= B_1 z^{2(n-1)} - a_{2n-1} B_1 z^{2n-3} + \\ &+ \left(B_n - a_{2(n-2)} B_1\right) z^{2(n-2)} + \\ &+ \left(B_{n-1} - a_{2n-3} B_1\right) z^{2n-5} + \\ &+ \left(B_{n-2} - a_{2(n-2)} B_1\right) z^{2(n-3)} + \\ &+ B_{n-3} z^{2(n-4)} + \ldots + B_3 z^2 + B_2 \end{aligned} \quad (26b)$$

From lemma 5 it follows that the strictly proper matrix (8) can be decomposed as follows

$$N(z) = P(z)Q(z) + d(z)G(z) \quad (27)$$

where

$$\begin{aligned} P(z) &= P_{2(n-1)} z^{2(n-1)} - P_{2n-3} z^{2n-3} + \\ &+ P_{2(n-2)} z^{2(n-2)} - \ldots + P_2 z^2 - P_1 z + P_0 \\ Q(z) &= Q_{2(n-1)} z^{2(n-1)} - Q_{2n-3} z^{2n-3} + \\ &+ Q_{2(n-2)} z^{2(n-2)} - \ldots + Q_2 z^2 - Q_1 z + Q_0 \\ G(z) &\in R^{p\times n}[z] \end{aligned} \quad (28)$$

(the set of polynomial matrices)

The polynomial matrices $P(z)$, $Q(z)$ and $G(z)$ of (27) can be computed by the use of the following procedure.

Procedure 1.

Step 1. Using elementary row and column operations perform the reduction

$$U(z)N(z)V(z) = p(z)\begin{bmatrix} 1 & r(z) \\ c(z) & M(z) \end{bmatrix}$$

where $U(z)$ and $V(z)$ are unimodular matrices of the elementary operations,
$p(z)$ is a polynomial, $r(z) \in R^{1\times(m-1)}[z]$,
$c(z) \in R^{p-1}[z]$ and $M(z) \in R^{(p-1)\times(m-1)}[z]$.

Step 2. Compute the matrices

$$P(z)=U^{-1}(z)p(z)\begin{bmatrix}1\\c(z)\end{bmatrix},Q(z)=[1 \quad r(z)]V^{-1}(z)$$

$$G(z)=U^{-1}(z)\begin{bmatrix}0 & 0\\0 & p(z)(M(z)-c(z)r(z))\end{bmatrix}V^{-1}(z)$$

Comparison of (26) and (28) yields

$$\begin{aligned}&P_{2(n-1)}=C_2,\quad P_{2n-3}=C_2a_{2n-1},\\&P_{2(n-2)}=C_3-a_{2(n-1)}C_2,\dots,P_0=\\&=C_1-a_2C_2-a_4C_3\\&Q_{2(n-1)}=B_1,\quad Q_{2n-3}=a_{2n-1}B_1,\\&Q_{2(n-2)}=B_n-a_{2(n-2)}B_1,\dots,Q_0=B_2\end{aligned}\tag{30}$$

Knowing the matrices P_k and Q_k for $k=0,1,...,2(n-1)$ we can find from (30) C_i and B_i, $i=1,\quad,n$ of the matrices C and B.

From (30) it follows that $B_i\in R_+^{1\times m}$ and $C_i\in R_+^{p}$ for $i=1,\quad,n$ if $P_k\in R_+^{p}$, $Q_k\in R_+^{1\times m}$ for $k=0,1,...,2(n-1)$ and $a_j\geq 0$ for $j=0,1,...,2n-1$

Therefore the following theorem has been proved.

Theorem 2. Let the transfer matrix (4) be normal. The positive realization problem has a solution if the following conditions are satisfied

(i) $T(\infty)=\lim_{z\to\infty}T(z)\in R^{p\times m}$ +

(ii) The coefficients a_k, $k=0,1,...,2n-1$ of the polynomial $d(z)$ are nonnegative.

(iii) The polynomial matrix $N(z)$ of (8) can be decomposed so that the polynomial $P(z)$ and $Q(z)$ (defined by (28)) have nonnegative coefficients matrices, i.e. $P_k\in R_+^{p}$, $Q_k\in R_+^{1\times m}$ for $k=0,1,...,2(n-1)$ and the relations (29) are satisfied.

If the conditions of theorem are satisfied then the positive realization (3) of $T(z)$ can be found by the use of the following procedure.

Procedure 2.

Step 1. Using (7) and (8) find D and the strictly proper rational matrix $T_{sp}(z)$

Step 2. Knowing the coefficients a_k, $k=0,1,...,2n-1$ of $d(z)$ find the matrices (12a) (or (12b), (12c), (12d))

Step 3. Using procedure 1 find the decomposition (27) of the polynomial matrix $N(z)$ of (8) and the coefficients matrices P_k and Q_k, of the polynomial matrices (28).

Step 4. Using (30) find B_i and C_i, $i=1,\quad,n$ and the matrices B and C.

Example. Find the positive realization (3) of the transfer matrix

$$T(z)=\frac{1}{z^5-z^4-2z^3-3z^2-2z-1}\times\begin{bmatrix}2z^5+7z^4+3z^3-4z-2, & z^5+2z^4-z\\z^5+10z^4+6z^3+4z^2+2z-1, & 4z^4+2z^3+3z^2+z\end{bmatrix}\tag{31}$$

Using the procedure 1 we obtain successively

Step 1. From (7) and (8) we have

$$D=\lim_{z\to\infty}T(z)=\begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\tag{32}$$

and

$$T_{sp}(z)=T(z)-D=\frac{N(z)}{d(z)}\tag{33}$$

where

$$N(z)=\begin{bmatrix}9z^4+7z^3+6z^2, & 3z^4+2z^3+3z^2+z+1\\11z^4+8z^3+7z^2+4z, & 4z^4+2z^3+3z^2+z\end{bmatrix};$$

$$d'(z)=z^5-z^4-2z^3-3z^2-2z-1$$

Step 2. Taking into account that $a_0=0$, $a_1=a_5=1$, $a_2=a_4=2$, $a_3=3$ and using (12a) we obtain

$$A_0=\begin{bmatrix}0 & 0 & 0\\1 & 0 & 3\\0 & 0 & 1\end{bmatrix},\quad A_1=\begin{bmatrix}0 & 0 & 1\\0 & 0 & 2\\0 & 1 & 2\end{bmatrix}\tag{34}$$

Step 3. Using the Procedure 1 we decompose the matrix $N(z)$ in the form (27) with

$$P(z)=\begin{bmatrix} z^4-z^3+1 \\ z^4-z^3+z^2 \end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}z^4-\begin{bmatrix}1\\1\end{bmatrix}z^3+$$
$$+\begin{bmatrix}0\\1\end{bmatrix}z^2+\begin{bmatrix}1\\0\end{bmatrix}=P_4z^4-P_3z^3+P_2z^2-P_1z+P_0$$
$$Q(z)=\begin{bmatrix}z^4-z^3+z^2-3z & z^2+1\end{bmatrix}=$$
$$=\begin{bmatrix}1 & 0\end{bmatrix}z^4-\begin{bmatrix}1 & 0\end{bmatrix}z^3+\begin{bmatrix}1 & 1\end{bmatrix}z^2+ \tag{35}$$
$$-\begin{bmatrix}3 & 0\end{bmatrix}z+\begin{bmatrix}0 & 1\end{bmatrix}=$$
$$=Q_4z^4-Q_3z^3+Q_2z^2-Q_1z+Q_0$$
$$G(z)=\begin{bmatrix}-z^2+z-3 & -1\\ -z^2+z-4 & -1\end{bmatrix}$$

Step 4. Using (30) and (35) we obtain

$$B_1=Q_4=\begin{bmatrix}1 & 0\end{bmatrix},\quad B_2=Q_0+a_2Q_4=\begin{bmatrix}2 & 1\end{bmatrix},$$
$$B_3=Q_2+a_4Q_4=\begin{bmatrix}3 & 1\end{bmatrix}$$
$$B=\begin{bmatrix}B_1\\B_2\\B_3\end{bmatrix}=\begin{bmatrix}1 & 0\\2 & 1\\3 & 1\end{bmatrix} \tag{36a}$$

and

$$C_1=P_0=\begin{bmatrix}1\\0\end{bmatrix},\quad C_2=P_4=\begin{bmatrix}1\\1\end{bmatrix},$$
$$C_3=P_2+a_4P_4=\begin{bmatrix}2\\3\end{bmatrix}$$
$$C=\begin{bmatrix}C_1 & C_2 & C_3\end{bmatrix}=\begin{bmatrix}1 & 1 & 2\\0 & 1 & 3\end{bmatrix} \tag{36b}$$

The desired positive minimal realization of (31) has the form (34), (36) and (32).

IV. CONCLUDING REMARKS

The realization problem for positive multivariable discrete-time systems with one time-delay has been formulated and solved. Canonical forms (12) of the system matrices A_0 and A_1 have been introduced. It has been shown that the pair (12) is cyclic. Conditions for the existence of positive minimal realization (3) of a proper rational matrix $T(z)$ have been established. A procedure for computation of a minimal positive realization of proper rational matrix has been presented and illustrated by an example. The considerations can be extended for multivariable discrete-time linear systems with many time-delays. An extension of the considerations for continuous-time linear systems with time-delays is also possible.

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