Hybrid BEM/FEM Forward model for Optical and Impedance Tomography

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Abstract— **The paper presents an hybrid approach to the forward problems important for Optical and Impedance Tomography. New formulation for the FEM is introduced in order to implement continuity and equilibrium conditions on the interface between BE and FE sub–regions. Selected 2D example results indicate some interesting tendency from the OT or EIT point of view.**

I. Introduction

There is a lot of discussion about the advantages and disadvantages of the BE method when compared to the FE one. Clearly there are certain applications where one technique is more suitable than the other. But for optical or impedance tomography problems combining both techniques in the same computer program, would be the most efficient way of modeling the human head. One could mentioned such a problems as the light point sources, scull or the void regions which could be treated by this technique, for example.

In order to take advantage of FEM and BEM, their coupling has been investigated extensively in several engineering fields, such as geomechanics [2], [3], solid mechanics [11], fracture mechanics [1] and electromagnetics [4], [8], [10], [6], [5]

There are several different method of coupling BEM and FEM [9], [8], [4]. The methods discussed in this paper are limited to the direct coupling of the BE and FE matrices when the boundary conditions on the interface between two subregions are imposed.

II. FEM-BEM coupling

This problem is closely related to the multi–region problem of the BE method such as presented in Fig. 1. The multi–region analysis has to fulfill continuity conditions along the interface line Γ_i between Ω_1 and Ω_2 regions. This results in the following two relationships

$$\Phi^{(1)}_{\Gamma_i} = \Phi^{(2)}_{\Gamma_i} \qquad \left.\frac{\partial \Phi^{(1)}}{\partial n}\right|_{\Gamma_i} = - \left.\frac{\partial \Phi^{(2)}}{\partial n}\right|_{\Gamma_i} \tag{1}$$

Let the sub–region Ω_1 be discretised by the Finite Elements and the Ω_2 by the Boundary Elements. Along the common interface, two conditions must be satisfied Eq.(1): continuity (the first one) and equilibrium (the second one), as it was in case of the multi–region BEM.

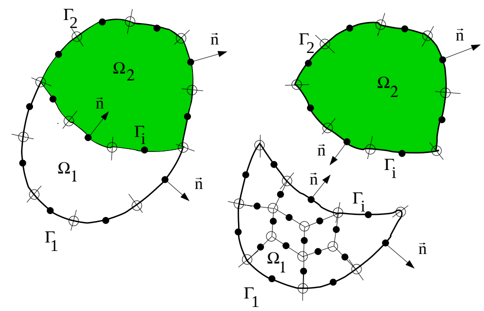

Fig. 1. The multi–region analysis

Continuity of the state function Φ can be maintained by using the same order of basis functions in both FE and BE formulations. Thus, if a three–noded isoparametric quadratic boundary element is used an equivalent finite element such as for example eight–noded quadrilateral quadratic element or six nodes isoparametric triangle has to be used for the finite element approximation.

The essentiality of the problem lies in the fact that the interpolation for the derivatives of the potential for the FEM lies one order lower than the order of the potential itself, whereas for the BEM formulation developed here, the interpolation functions has the same order not only for the potential but also for its derivatives.

Such unequal interpolation of the normal derivatives on the interface implants an error to the resulting system of equations as we could observe later.

Because along the interface the continuity and equilibrium conditions have to be fulfilled, for the FE approach we have to assume that on the interface we would have additional unknown–flux, expressed by Neumann boundary conditions. Normally to solve FE system the Neumann boundary conditions have to be known allowing us to solve the system of equations. So now the FE system of equations in its matrix form could be expressed as follows

$$\mathsf{A}^{(FE)}\Phi^{(FE)} = \mathsf{B}^{(FE)}\frac{\partial \Phi^{(FE)}}{\partial n} + \mathsf{F} \quad in \quad \Omega_{FE} \tag{2}$$

where $\Phi^{(FE)}$ and $\frac{\partial \Phi^{(FE)}}{\partial n}$ are column matrices containing the nodal values for the potential (photon density) in the whole sub–domain Ω_1 and its normal derivatives (current photon) on the interface

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Γ_i.

The corresponding boundary integral equation for the BEM sub–domain is given by

$$\mathsf{A}^{(BE)}\Phi = \mathsf{B}^{(BE)}\frac{\partial \Phi^{(BE)}}{\partial n} + \mathsf{q} \quad on \quad \Gamma_{BE} \quad (3)$$

where $\Phi^{(BE)}$ and $\frac{\partial \Phi^{(BE)}}{\partial n}$ are the nodal potentials (photon density) and theirs normal derivative vectors respectively.

Combining the Eq.(2), Eq.(3) and adding the interface conditions Eq.(1) one will get a system of equations ready for the solution (see Eq.(16).

III. Finite Element representation

The boundary–value problem for the FE sub–region is defined by the second–order differential equation [7]

$$-\frac{\partial}{\partial x}\left(D\frac{\partial \Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(D\frac{\partial \Phi}{\partial y}\right) + k^2\Phi = 0 \quad (4)$$

in conjunction with the boundary conditions (see Fig. 2):

$$\Phi = \Phi_0 \quad on \quad \Gamma \quad and \quad \frac{\partial \Phi}{\partial n} = \psi \quad on \quad \Gamma_i \quad (5)$$

The equivalent variational problem for the

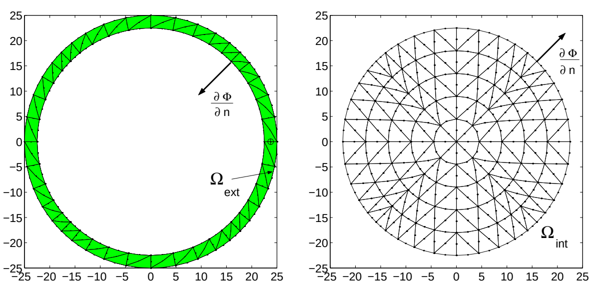

Fig. 2. Interface between two FE sub–regions

boundary–value problem defined above is given by

$$\delta F(\Phi) = 0 \qquad \Phi = \Phi_0 \qquad on \qquad \Gamma \quad (6)$$

where

$$F = \frac{D}{2}\int_{\Omega^{FE}}\left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + k^2\Phi^2\right]d\Omega + \int_{\Gamma_i}\Phi\psi d\Gamma_i \quad (7)$$

To discretize the functional (7), the FE sub–region is divided into M elements and interface boundary Γ_i (see Fig. 1) is broken into M_i segments. Usually, M is much larger than M_i. Within each area element, the field is expressed as

$$\Phi(x,y) = \sum_{i=1}^{6} N_i^e(x,y)\Phi_i^e = \{N^e\}^T\{\Phi^e\} = \{\Phi^e\}^T\{N^e\} \quad (8)$$

and on each line segment on the interface the field is expressed as

$$\Phi^i(x,y) = \sum_{i=1}^{3} N_i^i(x,y)\Phi_i^i = \{N^i\}^T\{\Phi^i\} = \{\Phi^i\}^T\{N^i\} \quad (9)$$

Assuming that the interface boundary Γ_i is a smooth contour, the normal derivative of the boundary field, which is ψ, is well defined at each interface node and therefore can also be expressed as

$$\psi^i(x,y) = \sum_{i=1}^{3} N_i^i(x,y)\psi_i^i = \{N^i\}^T\{\psi^i\} = \{\psi^i\}^T\{N^i\} \quad (10)$$

This is a weak point of this approach because the Φ and its normal derivative are approximated by the same shape functions. Results of such approach will be demonstrated later.

Substituting Eq.(8–10) into Eq.(7), we obtain

$$F = \frac{1}{2}\sum_{e=1}^{M}\{\Phi^e\}^T[A^e]\{\Phi^e\} + \sum_{i=1}^{M_i}\{\Phi^i\}^T[B^i]\{\psi^i\} \quad (11)$$

where

$$[A^e] = \int_{\Omega^e} D\left[\left\{\frac{\partial N^e}{\partial x}\right\}\left\{\frac{\partial N^e}{\partial x}\right\}^T + \left\{\frac{\partial N^e}{\partial y}\right\}\left\{\frac{\partial N^e}{\partial y}\right\}^T\right]dxdy - \int_{\Omega^e} k_0^2\{N^e\}\{N^e\}^T dxdy$$

and

$$[B^i] = \int_{\Gamma_i}\{N^i\}\{N^i\}^T d\Gamma_i \quad (12)$$

Provided that the element length of the interface is small, the Jacobian of transformation to local coordinate system may be assumed constant and taken out of the integral sign in Eq.(12) without causing significant errors. Therefore, by substituting the explicit expressions for the shape functions, it is easy to perform the indicated integrations analytically. So the entries of matrix $[B^i]$ in case of the quadrilateral three nodes isoparametric elements of the interface in local coordinate system are defined by

$$[B^i] = \begin{bmatrix} N_1^iN_1^i & N_1^iN_2^i & N_1^iN_3^i \\ N_2^iN_1^i & N_2^iN_2^i & N_2^iN_3^i \\ N_3^iN_1^i & N_3^iN_2^i & N_3^iN_3^i \end{bmatrix} J(\xi) = \begin{bmatrix} \frac{4}{15} & \frac{2}{15} & -\frac{1}{15} \\ \frac{2}{15} & \frac{16}{15} & -\frac{2}{15} \\ -\frac{1}{15} & \frac{2}{15} & -\frac{4}{15} \end{bmatrix} J(\xi) \quad (13)$$

Than, performing the assembly Eq.(11) can be written as

$$F = \frac{1}{2}\{\Phi\}^T\left[\mathsf{A}^{(FE)}\right]\{\Phi\} + \{\Phi\}^T\left[\mathsf{B}^{(FE)}\right]\{\psi\} \quad (14)$$

where $\mathsf{A}^{(FE)}$ is an $N \times N$ square matrix, $\mathsf{B}^{(FE)}$ is an $N \times M_i$ rectangular matrix, Φ is a column vector representing the nodal values of field intensity and ψ is a column vector representing the nodal values of $\frac{\partial \Phi}{\partial n}$ on the M_i nodes of the interface. Differentiating F with respect to each nodal field and equating the resulting expression to zero yields a system of linear equations

$$\mathsf{A}^{(FE)}\Phi^{(\mathsf{FE})} + \mathsf{B}^{(FE)}\frac{\partial\Phi^{(FE)}}{\partial n} = 0 \tag{15}$$

Now the hybrid system of equations can be formed in the following way

$$\begin{aligned}&\left[\begin{array}{cc} \mathsf{A}^{(FE)} & 0 \\ 0 & \mathsf{A}^{(BE)} \end{array}\right]\left[\begin{array}{c} \Phi^{(FE)} \\ \Phi^{(BE)} \end{array}\right] = \\ &\left[\begin{array}{cc} -\mathsf{B}^{(FE)} & 0 \\ 0 & \mathsf{B}^{(BE)} \end{array}\right]\left[\begin{array}{c} \frac{\partial\Phi^{(FE)}}{\partial n} \\ \frac{\partial\Phi^{(BE)}}{\partial n} \end{array}\right] + \left[\begin{array}{c} 0 \\ \mathsf{q} \end{array}\right]\end{aligned} \tag{16}$$

The resulting matrices have to be rearranged to accommodate the continuity and equilibrium conditions (Eq.(1)) as well as the prescribed boundary conditions. As a result we will get the matrix with the following structure The matrix is unsymmetrical with much bigger bandwidth with two additional group of non–zero elements caused by the interface between FE and BE sub–domains. Rearrangement of resulting matrix is very similar to that required in multi–region BEM problems.

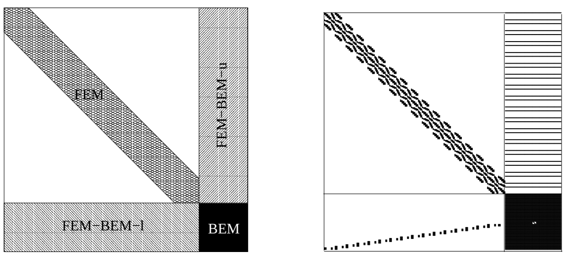

Fig. 3. The FEM–BEM matrix structure

IV. Numerical results

The coupling BEM and FEM is well established problem in the literature. But we are interested in a particular geometry of the regions and their approximations by BEM/FEM. Precision of the solutions is extremely sensitive on geometry configuration and the boundary conditions. Such a problems will be illustrated by 2D carefully selected examples which in some sense are similar to cross sections of the human head.

A. FE square immersed in BE region

Let us consider simple but interesting example when the FE region is immersed in the BE subregion (see Fig.4) and The same numerical example but the Finite Element sub–region will be discretised by isoparametric quadratic triangular elements (see Fig. 4). We can see that discretization has a little influence on the precision of the solution. This is a very good news as we need to create a model with as few nodes as possible.

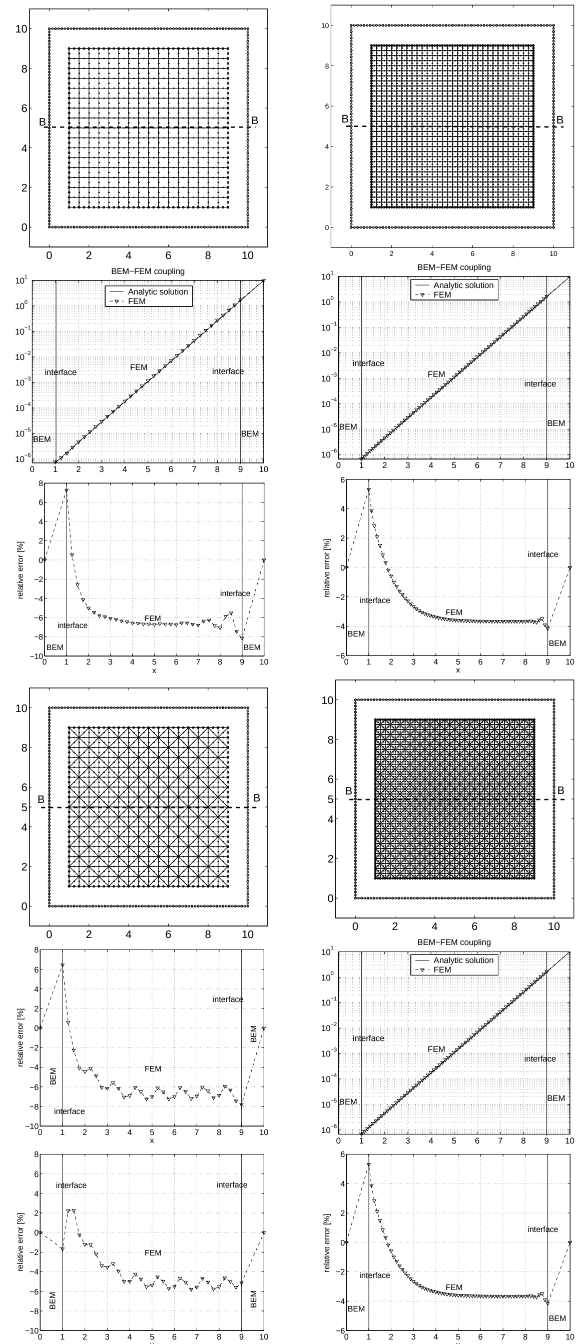

Fig. 4. Rectangular FEM region immersed in BEM: discretizations and solutions for different elements and grid densities

B. Concentric circles

In order to present the sensitivity of the results on boundary conditions the following example presented in two first columns in Fig. 5 was considered. We can see that interface error is propagating inside the FE sub–region, what is highly undesired behavior. In order to eliminate possibility of the software bug we have solved two FE sub–regions coupled by interface conditions expressed by Eq. (1). As we can see in third and fourth columns Fig. 5 the error along the radius is almost equal to zero except of the interface points. On this basis we are entitled

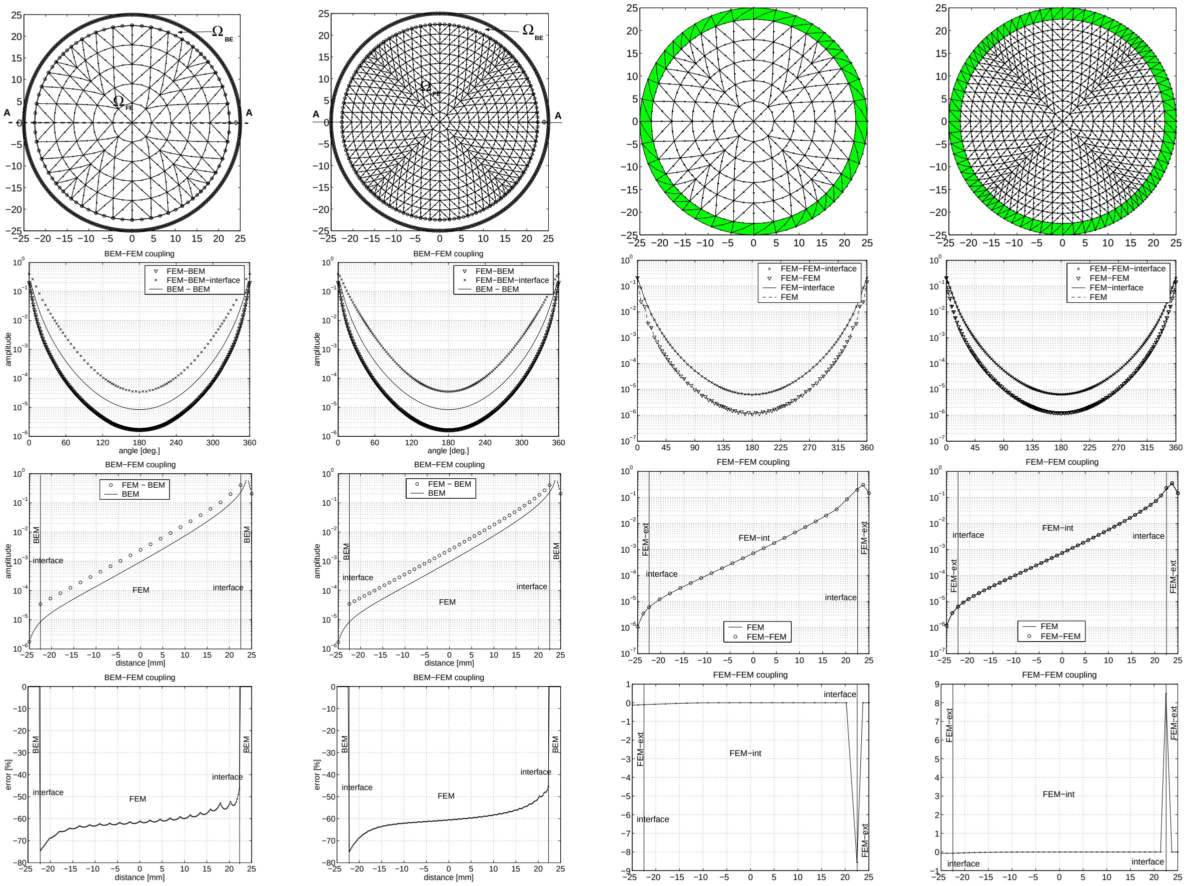

Fig. 5. Circular region: hybrid FEM-BEM solution: coarse discretization (first column) and dense one (second column) and FEM-FEM coupling: coarse discretisation (third column) and dense (fourth column)

to conclude that software is constructed correctly.

V. Conclusion

The application of the hybrid approach was presented in this paper. Unfortunately on the interface for this formulation of BEM high error occurred on the interface. There are at least two ways to avoid such errors which are propagated inside of the FE sub–region. The first approach is to reformulate the normal derivative approximation and the second to apply the Gallerkin formulation of BEM. Both approaches will be investigated in the future.

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