Nonlocal Boundary Conditions in 2D Regions with Clear Layer Gap

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Abstract— **The paper presents an analytical solution of the clear region immersed in the highly scattering one. This problem is very important in Optical Tomography. The analytical solution is an excellent benchmark problem so we can compare the numerical solution like FEM or BEM with the analytical one presented in this paper.**

I. Introduction

For light transport the Boltzman equation is approximated by the diffusion equation for simplicity [1]. Consider the diffusion equation in frequency domain.

$$\nabla \cdot \nabla \Phi(\mathbf{r};\omega) - \delta^2 \Phi(\mathbf{r};\omega) = -\frac{Q_0(\mathbf{r};\omega)}{D(\mathbf{r})} \qquad \mathbf{r} \in \Omega \quad (1)$$

where $\delta = \sqrt{\frac{\mu_a(\mathbf{r})}{D(\mathbf{r})} - \frac{j\omega}{cD(\mathbf{r})}}$, diffusion parameter $D(\mathbf{r}) = \frac{1}{2(\mu_a + \mu_s{}')}$ and $\mu_a, \mu_s^{'}$ are absorbing and reduced scattering coefficients respectively, c – speed of light.

Approximation of Boltzman equation by Eq.(1) is valid only in case of highly scattering regions. That means the CSF layers (clear layer/low scattering layer) could not be described by diffusion equation.

In such cases frequently occurred in Optical Tomography the clear layer should be replaced by the non–local boundary conditions [1], [4], [5], [6], [8], [11].

Let us consider the simple 2D example (see Fig. 1), having the analytical solution [6]. The dimensions of the region are typical for Optical Tomography baby's simplified head model and have the following values: $r_1 = 25.0mm$, $r_2 = 22.5mm$ and $r_3 = 19.5mm$. The strongly scattering regions data were assumed: $\mu_a = 0.1mm^{-1}$ and $\mu_s' = 1.0mm^{-1}$ and in non–scattering region $\mu_a = 0.1mm^{-1}$ (see for example [2]). For the sake of simplicity, the unit input flux on the outermost boundary was imposed and a steady state was considered. Geometry of the region and imposed boundary conditions reduce this problem to 1D.

II. Analytical solution for diffusive boundary conditions

Diffusive nonlocal boundary conditions used in FEM introduced by Arridge in [8] and next used by many others mainly in the FEM code [3], [6], [7], [8]. Let us consider 2D region shown in Fig. 1. If

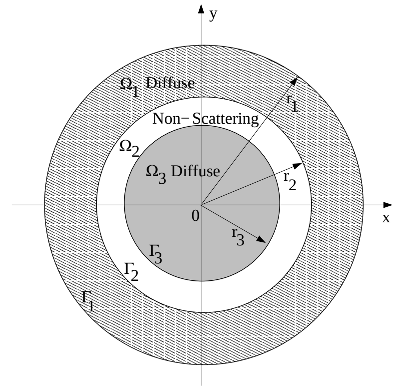

Fig. 1. 2D region with a non–scattering gap

we assume that incoming flux is constant, than the boundary conditions on the outermost boundary Γ_1 are

$$-\left(\gamma\phi + \frac{D}{2}\frac{\partial\phi}{\partial n}\right)\bigg|_{\Gamma_1} = -1 \qquad (2)$$

Due to the symmetry of the region and constant boundary conditions solution will take the form

$$\phi(r,\Theta) = c^I I_o(\delta r) + c^K K_o(\delta r) \qquad (3)$$

Where I_o and K_o means the zero order Bessel functions of the first and second kind respectively. For a diffusive region Ω_1

$$\begin{aligned} \phi_1(r,\Theta) &= c_1^I I_o(\delta r) + c_1^K K_o(\delta r) \\ \frac{\partial \phi_1(r,\Theta)}{\partial n} &= c_1^I \delta I_1(\delta r) - c_1^K \delta K_1(\delta r) \end{aligned} \qquad (4)$$

Due to the singularity of the K_o function when r tends to zero, solution for a diffusive region Ω_3 become

$$\phi_3(r,\Theta) = c_3^I I_o(\delta r) \qquad \frac{\partial \phi_3(r,\Theta)}{\partial n} = c_3^I \delta I_1(\delta r) \quad (5)$$

As we can see the solution does not depend on the angle Θ. We have three unknown coefficients which could be calculated by implementing boundary conditions on the Γ_1, Γ_2 and Γ_3 as follows:

for the boundary Γ_1

$$\begin{aligned} 2\gamma\phi_{\Gamma_1}(r) + D\frac{\partial \phi_{\Gamma_1}(r)}{\partial n} = 2 \\ c_1^I(2\gamma I_o(\delta r_1) + D\delta I_1(\delta r_1) + c_1^K(2\gamma K_o(\delta r_1) + \\ -D\delta K_1(\delta r_1)) = 2 \end{aligned} \qquad (6)$$

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for the boundary $\Gamma_2 - \mathbf{r} \in \Gamma_2$ and $\mathbf{r}' \in \Gamma_2 \cup \Gamma_3$

$$\phi_{\Gamma_2}(r) = -\left(-\alpha D \frac{\partial \phi_{\Gamma_2}(r)}{\partial n}\right) + \frac{1}{\pi\alpha} \int\limits_{\Gamma_i=\Gamma_2\cup\Gamma_3} \phi_{\Gamma_i}(r')\mathcal{G}(|\mathbf{r}-\mathbf{r}'|)d\Gamma_i \quad (7)$$

Due to imposed boundary conditions, ϕ_{Γ_i} is constant so one can write:

$$\phi_{\Gamma_2}(r) = \alpha D \frac{\partial \phi_{\Gamma_2}(r)}{\partial n} + \frac{1}{\pi\alpha}\phi_{\Gamma_2}(r')\int\limits_{\Gamma_2} \mathcal{G}(|\mathbf{r}-\mathbf{r}'|)d\Gamma_2 + \frac{1}{\pi\alpha}\phi_{\Gamma_3}(r')\int\limits_{\Gamma_3} \mathcal{G}(|\mathbf{r}-\mathbf{r}'|)d\Gamma_3 \quad (8)$$

where $\alpha = 1$ in our case.

for the boundary $\Gamma_3 - \mathbf{r} \in \Gamma_3$ and $\mathbf{r}' \in \Gamma_3 \cup \Gamma_2$

$$\phi_{\Gamma_3}(r) = -\alpha D \frac{\partial \phi_{\Gamma_3}(r)}{\partial n} + + \frac{1}{\pi\alpha} \int\limits_{\Gamma_i=\Gamma_2\cup\Gamma_3} \phi_{\Gamma_i}(r')\mathcal{G}(|\mathbf{r}-\mathbf{r}'|)d\Gamma_i \quad (9)$$

and finally:

$$\phi_{\Gamma_3}(r) = -\alpha D \frac{\partial \phi_{\Gamma_3}(r)}{\partial n} + \frac{1}{\pi\alpha}\phi_{\Gamma_2}(r')\int\limits_{\Gamma_2} \mathcal{G}(|\mathbf{r}-\mathbf{r}'|)d\Gamma_2 \quad (10)$$

where the operator $\mathcal{G}(|\mathbf{r}-\mathbf{r}'|)$ is the radiosity kernel [9], representing diffuse-diffuse propagation of light in free space and is equal to zero when the vector $\mathbf{r}' \in \Gamma_3$. Points on the most inner circle Γ_3 are not visible for itself.

Implementing Eq.(4) and Eq.(5) into Eq.(7) and into Eq.(9) we will get:

$$\begin{aligned} c_1^I I_o(\delta r_3) &= -\alpha D c_1^I \delta I_1(\delta r_3) + g_1 c_2^I I_o(\delta r_2) + g_1 c_2^K K_o(\delta r_2) \\ c_3^I\left[I_o(\delta r_2) - \alpha D\delta I_1(\delta r_2)\right] &+ c_3^K\left[K_o(\delta r_2) + \alpha D\delta K_1(\delta r_2)\right] = \\ & g_3\left[c_3^I I_o(\delta r_2) + c_3^K K_o(\delta r_2)\right] + g_2 c_1^I I_o(\delta r_3) \end{aligned} \quad (11)$$

Adding the Eq.(6) and doing some math we will get a system of linear equations which alow us to calculate unknown coefficients c_1^I, c_1^K and c_3^I.

$$\begin{aligned} & c_1^I(2\gamma I_o(\delta r_1) + D\delta I_1(\delta r_1) + c_1^K(2\gamma K_o(\delta r_1) - D\delta K_1(\delta r_1)) = 2 \\ & -c_1^I g_1 I_o(\delta r_2) - c_1^K g_1 K_o(\delta r_2) + c_3^I\left[I_o(\delta r_3) + \alpha D\delta I_1(\delta r_3)\right] = 0 \\ & c_1^I\left[(1-g_3)I_o(\delta r_2) - \alpha D\delta I_1(\delta r_2)\right] + c_1^K\left[(1-g_3)K_o(\delta r_2) + \alpha D\delta K_1(\delta r_2)\right] - c_3^I g_2 I_o(\delta r_3) = 0 \end{aligned} \quad (12)$$

III. Analytical solution for P_1 boundary conditions

P_1 boundary conditions were suggested by Rippol in [9] and used for integral formulation of OT by others [11], [10], [12], but till now no comparative analysis between diffusive and P_1 nonlocal boundary conditions does exists. Now, using the 2D benchmark presented in Fig. 1, we will make the comparison and we would be interested what difference are caused by using Diffusive (FEM) or P_1 (BEM) boundary conditions. For the boundary Γ_1 the Robin boundary conditions remain the same as for diffusive boundary conditions. For the boundary $\Gamma_2 - \mathbf{r} \in \Gamma_2$ and $\mathbf{r}' \in \Gamma_2 \cup \Gamma_3$ and boundary $\Gamma_3 - \mathbf{r} \in \Gamma_3$ and $\mathbf{r}' \in \Gamma_2$, the nonlocal P_1 boundary conditions are imposed:

$$\begin{aligned} \phi_{\Gamma_2}(r) &= \alpha D \frac{\partial \phi_{\Gamma_2}(r)}{\partial n} + \\ & \frac{1}{\pi}\left[\phi_{\Gamma_2}(r') + D\frac{R_J}{R_U}\frac{\partial \phi_{\Gamma_2}(r')}{\partial n}\right]\int\limits_{\Gamma_2} \mathcal{G}(|\mathbf{r}-\mathbf{r}'|)d\Gamma_2 + \\ & \frac{1}{\pi}\left[\phi_{\Gamma_3}(r') - D\frac{R_J}{R_U}\frac{\partial \phi_{\Gamma_3}(r')}{\partial n}\right]\int\limits_{\Gamma_3} \mathcal{G}(|\mathbf{r}-\mathbf{r}'|)d\Gamma_3 \\ \phi_{\Gamma_3}(r) &= -\alpha D \frac{\partial \phi_{\Gamma_3}(r)}{\partial n} + \\ & \frac{1}{\pi}\left[\phi_{\Gamma_2}(r') + D\frac{R_J}{R_U}\frac{\partial \phi_{\Gamma_2}(r')}{\partial n}\right]\int\limits_{\Gamma_2} \mathcal{G}(|\mathbf{r}-\mathbf{r}'|)d\Gamma_2 \end{aligned} \quad (13)$$

Finally we will get a system of linear equations for c_1^I, c_1^K and c_3^I unknowns.

$$\begin{aligned} & c_1^I(2\gamma I_o(\delta r_1) + D\delta I_1(\delta r_1) + c_1^K(2\gamma K_o(\delta r_1) - D\delta K_1(\delta r_1)) = 2 \\ & -c_1^I g_1\left[I_o(\delta r_2) + D\frac{R_J}{R_U}\delta I_1(\delta r_2)\right] - c_1^K g_1\left[K_o(\delta r_2) - D\frac{R_J}{R_U}\delta K_1(\delta r_2)\right] + c_3^I\left[I_o(\delta r_3) + \alpha D\delta I_1(\delta r_3)\right] = 0 \\ & c_1^I\left[(1-g_3)I_o(\delta r_2) - D\delta\left(\alpha + g_3\frac{R_J}{R_U}\right)I_1(\delta r_2)\right] + c_1^K\left[(1-g_3)K_o(\delta r_2) + D\delta\left(\alpha + g_3\frac{R_J}{R_U}\right)K_1(\delta r_2)\right] - \\ & c_3^I g_2\left[I_o(\delta r_3) - D\frac{R_J}{R_U}\delta I_1(\delta r_3)\right] = 0 \end{aligned} \quad (14)$$

Solution of the system of algebraic equations Eq. (12) and Eq. (14) will provide the coefficients c_1^I, c_1^K and c_3^I for the analytical solution. But first we have to know haw to calculate operators g_1, g_2 and g_3. Those operators depend on visibility function which is included into radiosity kernel $\mathcal{G}$ so we can call them visibility operators.

IV. Visibility operators

Let us consider the case where $\mathbf{r} \in \Gamma_3$ and $\mathbf{r}' \in \Gamma_2$ (see Fig. 2)

$$\begin{aligned} cosU_o &= \frac{r_3}{r_2} \\ cos\Theta &= \hat{n}\cdot\frac{(\mathbf{r}'-\mathbf{r})}{(|\mathbf{r}'-\mathbf{r}|)} \\ cos\Theta' &= \hat{n}'\cdot\frac{(\mathbf{r}-\mathbf{r}')}{(|\mathbf{r}-\mathbf{r}'|)} \end{aligned} \quad (15)$$

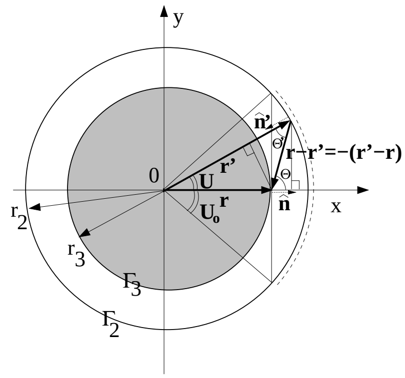

Fig. 2. Visible part of boundary Γ_2 when $\mathbf{r} \in \Gamma_3$

because

$$(|\mathbf{r}' - \mathbf{r}|) = (|\mathbf{r} - \mathbf{r}'|) = (r_2^2 - 2r_2r_3cosU + r_3^2)^{1/2}$$
$$cos\Theta = \frac{r_2cosU - r_3}{(r_2^2 - 2r_2r_3cosU + r_3^2)^{1/2}}$$
$$cos\Theta' = \frac{r_2 - r_3cosU}{(r_2^2 - 2r_2r_3cosU + r_3^2)^{1/2}}$$

$$g_1 = \int_{\Gamma_2} \mathcal{G}(|\mathbf{r} - \mathbf{r}'|)d\Gamma_2 =$$
$$2r_2 \int_0^{arccos(\frac{r_3}{r_2})} \nu(\mathbf{r}')\frac{cos\Theta cos\Theta'}{|\mathbf{r} - \mathbf{r}'|}\frac{1}{2}e^{-\mu_a|\mathbf{r}-\mathbf{r}'|}dU =$$
$$2r_2 \int_0^{arccos(\frac{r_3}{r_2})} \frac{r_2^2cosU - r_2r_3(1 + cos^2U) + r_3^2cosU}{(r_2^2 - 2r_2r_3cosU + r_3^2)^{3/2}} \frac{1}{2}e^{-\mu_a(r_2^2-2r_2r_3cosU+r_3^2)^{1/2}}dU$$

Let us consider the case where $\mathbf{r} \in \Gamma_2$ and $\mathbf{r}' \in \Gamma_2$ (see Fig. 3)

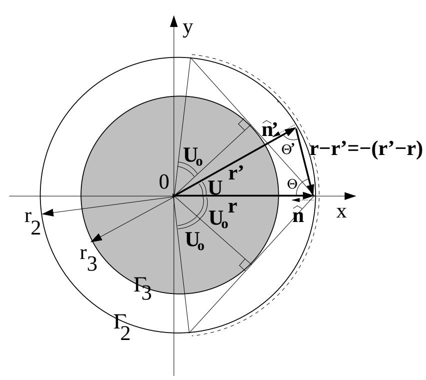

Fig. 3. Visible part of boundary Γ_2 when $\mathbf{r} \in \Gamma_2$

$$(|\mathbf{r}' - \mathbf{r}|) = (|\mathbf{r} - \mathbf{r}'|) =$$
$$2r_2sin(\frac{U}{2}) = r_2(2(1 - cosU))^{1/2}$$
$$cos\Theta = \frac{|\mathbf{r}' - \mathbf{r}|}{2r_2} \quad (16)$$
$$cos\Theta' = \frac{|\mathbf{r} - \mathbf{r}'|}{2r_2}$$

$$g_3 = \int_{\Gamma_2} \mathcal{G}(|\mathbf{r} - \mathbf{r}'|)d\Gamma_2 =$$
$$\frac{2r_2}{2\sqrt{2}} \int_0^{2arccos(\frac{r_3}{r_2})} \nu(\mathbf{r}')\frac{cos\Theta cos\Theta'}{|\mathbf{r} - \mathbf{r}'|}\frac{1}{2}e^{-\mu_a|\mathbf{r}-\mathbf{r}'|}dU =$$
$$\frac{1}{2\sqrt{2}} \int_0^{2arccos(\frac{r_3}{r_2})} (1 - cosU)^{1/2}e^{-\mu_a r_2(2(1-cosU))^{1/2}}dU$$

And finally let us consider the case where $\mathbf{r} \in \Gamma_2$ and $\mathbf{r}' \in \Gamma_3$ (see Fig. 4)

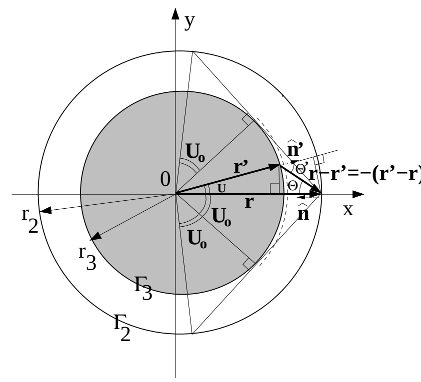

Fig. 4. Visible part of boundary Γ_3 when $\mathbf{r} \in \Gamma_2$

$$(|\mathbf{r}' - \mathbf{r}|) = (|\mathbf{r} - \mathbf{r}'|) = (r_2^2 - 2r_2r_3cosU + r_3^2)^{1/2}$$
$$cos\Theta = \frac{r_2 - r_3cosU}{(r_2^2 - 2r_2r_3cosU + r_3^2)^{1/2}}$$
$$cos\Theta' = \frac{r_2cosU - r_3}{(r_2^2 - 2r_2r_3cosU + r_3^2)^{1/2}}$$

$$g_2 = \int_{\Gamma_3} \mathcal{G}(|\mathbf{r} - \mathbf{r}'|)d\Gamma_3 =$$
$$2r_3 \int_0^{arccos(\frac{r_3}{r_2})} \nu(\mathbf{r}')\frac{cos\Theta cos\Theta'}{|\mathbf{r} - \mathbf{r}'|}\frac{1}{2}e^{-\mu_a|\mathbf{r}-\mathbf{r}'|}dU =$$
$$2r_3 \int_0^{arccos(\frac{r_3}{r_2})} \frac{r_2^2cosU - r_2r_3(1 + cos^2U) + r_3^2cosU}{(r_2^2 - 2r_2r_3cosU + r_3^2)^{3/2}} \frac{1}{2}e^{-\mu_a(r_2^2-2r_2r_3cosU+r_3^2)^{1/2}}dU$$

$$g_2 = \frac{r_3}{r_2}g_1 \quad (17)$$

V. Numerical results

In order to present the sensitivity of the results on the geometry of the domain and optical parameters two examples were considered. The first one for the region with normalized dimensions: $r_1 = 1.0, r_2 = 0.8$ and $r_3 = 0.5$. The strongly scattering regions data were assumed: $\mu_a = 0.5$ and $\mu_s^{'} = 50.0$ and in non–scattering region $\mu_a = 0.25$ (see for example [6]). This kind of data values reflects a neonatal brain model of diameter $100mm$. The second example is typical 2D benchmark for OT as was mentioned in the introduction. Comparison of the solution for the internal field distribution along the radius of the domain is shown in Fig. 6 and in Fig. 8. As a reference to the solution with a clear layer, the solution for the homogeneous diffusive region is shown in the same figure.

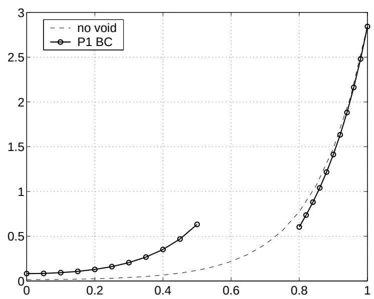

Fig. 5. Normalized region for P_1 B.C.

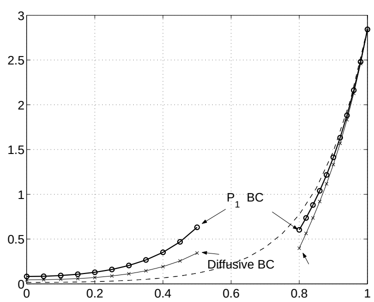

Fig. 6. Comparison with diffusive B.C.

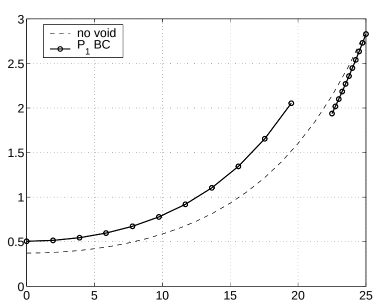

Fig. 7. The second example for P_1 B.C.

VI. Conclusion

The application of the analytical method to the regions containing the non–scattering inclusions is

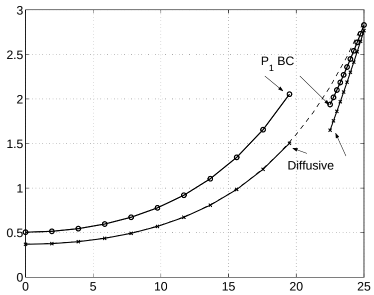

Fig. 8. Comparison with diffusive B.C.

presented in this paper. Two different non–local boundary conditions have been applied and the solutions were compared. Achieved results indicate a great influence the boundary conditions and the geometry of the region under consideration on the quality of the solution. Still there is an open question which one better reflect the reality. This will be the next question which we would like to answer.

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