Speeding up Waveform Relaxation Algorithms

Marek Ossowski, Andrzej Kuczyński

Abstract—**The paper deals with a problem of improving efficiency of the waveform relaxation (WR) algorithms for linear and nonlinear dynamic circuits. An algorithm for solving differential equations using multisplitting concept and overlapping procedures, was formulated in a way that provides the convergence of iteration process. In order to split the circuit into independent parts, the chosen branches are cut off and appropriate sources, voltage or current, are inserted instead of eliminated links. Voltages and currents of these sources form an additional set of algebraic variables, carrying information between separated parts. The conditions that are to ensure the convergence of iteration process were considered. The special circuit splitting and overlapping, guarantee fulfilment of the convergence conditions.**

I. Introduction

THE waveform relaxation (WR) methods, introduced by Lelarasmee [7] and developed during last ten years by many authors, are still actractive alternative to the incremental-time (IT) direct methods. They enable us to significantly decrease the size of analysed circuits and are suitable, in natural way, to be implemented in parallel-processor computer systems (parallel programming). The main advantage of this method is the possibility of using different step size for different subsystems: subcircuits only with "slow" components can be integrated with larger step size than those with "fast" components. Slow convergence, a result of a strong coupling between subcircuits, seems to be a major disadvantage of the investigated methods. Moreover, depending on the size of time window, much more memory is needed to store discrete solutions of the previous iteration for every subsystem. WR algorithms become worth attention, if the number of iterations, ensuring the predefined accuracy of the solution, is small enough. The problem with improving efficiency without loosing exactness seems to be hard to solve for general circuits. Some authors [14], [15], used the Succesive Overrelaxation Algotithms with autoadaptive choice of the relaxation factor. Today, multisplitting techniques together with overlapping methods [11], [12], [13] are more aplicable.

The main purpose of this paper is to present WR algorithm for solving nonlinear differential circuits that preserves convergence of iterations and guarantees much better efficiency than standard techniques. Considerations related to convergence and efficiency of the proposed method are illustrated by the representative example. Its comparison to standard algorithms is also included.

Authors are with the Institute of the Electrotechnics, Measuremants and Material Science, Technical University of Lodz, ul.Stefanowskiego 18/22, 00-000 Lodz, Poland, e-mail: akuczyn@mail.p.lodz.pl, mosso@mail.p.lodz.pl.

II. Fundamentls of the method

A. Circuit equations

We will investigate the circuit decomposed into N mutually connected subcircuits. An exemplary splitting for N=4 is shown in figure 1.

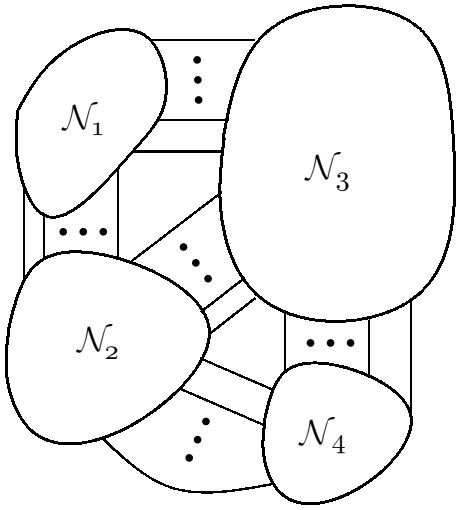

Fig. 1. Circuit decomposed into 4 subcircuits

To simplify further explanations, let us consider the circuits splitted into two subcircuits, labeled $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively, connected by three wires (see figure 2).

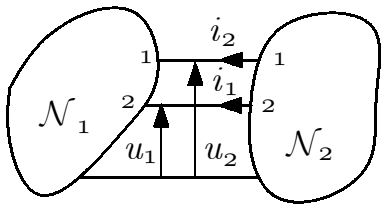

Fig. 2. Connections between two subcircuits

Introducing additional sources, instead of the removed connections, we have two systems presented in figure 3. Circuits from figures 2 and 3 are equivalent, if the following relations are fulfilled:

$$\hat{\hat{\mathrm{i}}}_{11} = \mathrm{i}_{21} \,, \qquad \hat{\hat{\mathrm{u}}}_{12} = \mathrm{u}_{22} \,, \tag{1}$$

$$\hat{\hat{\mathrm{i}}}_{22} = -\mathrm{i}_{12} \,, \qquad \hat{\hat{\mathrm{u}}}_{21} = -\mathrm{u}_{11} \,. \tag{2}$$

Let $\hat{\hat{\mathbf{s}}}_1$ and $\hat{\hat{\mathbf{s}}}_2$ be the vectors of additional sources

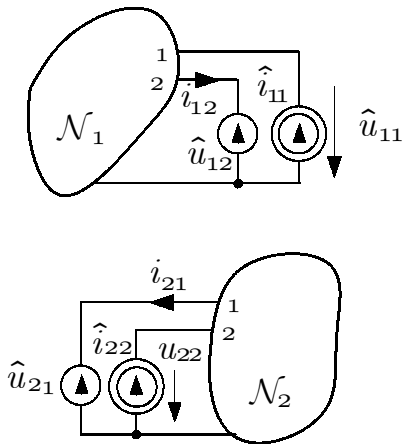

Fig. 3. Circuits with additional sources

for circuits $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively:

$$\hat{\hat{\mathbf{s}}}_1(t) = \begin{bmatrix} \hat{\hat{i}}_{11}(t) \\ \hat{\hat{u}}_{12}(t) \end{bmatrix} = \begin{bmatrix} i_{21}(t) \\ u_{22}(t) \end{bmatrix}, \quad (3)$$

$$\hat{\hat{\mathbf{s}}}_2(t) = \begin{bmatrix} \hat{\hat{u}}_{21}(t) \\ \hat{\hat{i}}_{22}(t) \end{bmatrix} = \begin{bmatrix} -u_{11}(t) \\ -i_{12}(t) \end{bmatrix}. \quad (4)$$

Circuits $\mathcal{N}_1$ and $\mathcal{N}_2$ are described by equations:

$$\mathbf{M}_1 \dot{\mathbf{x}}_1(t) + \mathbf{A}_1 \mathbf{x}_1(t) = \mathbf{D}_1 \mathbf{s}_1(t) - \mathbf{G}_1' \hat{\hat{\mathbf{s}}}_1(t), \quad (5)$$

$$\mathbf{M}_2 \dot{\mathbf{x}}_2(t) + \mathbf{A}_2 \mathbf{x}_2(t) = \mathbf{D}_2 \mathbf{s}_2(t) - \mathbf{G}_2' \hat{\hat{\mathbf{s}}}_2(t), \quad (6)$$

where $\mathbf{M}_1 \in \mathbf{R}^{N_1 \times N_1}$, $\mathbf{M}_2 \in \mathbf{R}^{N_2 \times N_2}$ (matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are nonsingular by assumption), $\mathbf{A}_1 \in \mathbf{R}^{N_1 \times N_1}$, $\mathbf{A}_2 \in \mathbf{R}^{N_2 \times N_2}$, $\mathbf{D}_1 \in \mathbf{R}^{N_1 \times Z_1}$, $\mathbf{D}_2 \in \mathbf{R}^{N_2 \times Z_2}$, $\mathbf{G}_1' \in \mathbf{R}^{N_1 \times \hat{Z}_1}$, $\mathbf{G}_2' \in \mathbf{R}^{N_2 \times \hat{Z}_2}$, vectors $\mathbf{x}_1 \in \mathbf{R}^{N_1}$ i $\mathbf{x}_2 \in \mathbf{R}^{N_2}$ denote state variables, vectors $\mathbf{s}_1 \in \mathbf{R}^{Z_1}$ and $\mathbf{s}_2 \in \mathbf{R}^{Z_2}$ contain independent sources placed inside individual circuits, vectors $\hat{\hat{\mathbf{s}}}_1 \in \mathbf{R}^{\hat{Z}_1}$ and $\hat{\hat{\mathbf{s}}}_2 \in \mathbf{R}^{\hat{Z}_2}$ contain additional sources.

It follows from the equations (3) and (4) that the values of additional independent sources can be defined by the relations:

$$\hat{\hat{\mathbf{s}}}_1(t) = -\mathbf{B}_2 \mathbf{x}_2(t) + \mathbf{C}_2 \mathbf{s}_2(t) = \mathbf{N}_2 \hat{\mathbf{s}}_2(t), \quad (7)$$

$$\hat{\hat{\mathbf{s}}}_2(t) = -\mathbf{B}_1 \mathbf{x}_1(t) + \mathbf{C}_1 \mathbf{s}_1(t) = \mathbf{N}_1 \hat{\mathbf{s}}_1(t), \quad (8)$$

where $\mathbf{B}_2 \in \mathbf{R}^{\hat{Z}_1 \times N_2}$, $\mathbf{B}_1 \in \mathbf{R}^{\hat{Z}_2 \times N_1}$, $\mathbf{C}_2 \in \mathbf{R}^{\hat{Z}_1 \times Z_2}$, $\mathbf{C}_1 \in \mathbf{R}^{\hat{Z}_2 \times Z_1}$ $\mathbf{N}_2 = \mathrm{diag}(1, \ldots, 1) \in \mathbf{R}^{\hat{Z}_2 \times \hat{Z}_2}$ $\mathbf{N}_1 = \mathrm{diag}(1, \ldots, 1) \in \mathbf{R}^{\hat{Z}_1 \times \hat{Z}_1}$.

Matrix description which results from the above relations is of the form:

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \\ \dot{\hat{\mathbf{s}}}_1 \\ \dot{\hat{\mathbf{s}}}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{G}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{0} & \mathbf{G}_2 \\ \mathbf{B}_1 & \mathbf{0} & \mathbf{N}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \mathbf{0} & \mathbf{N}_2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \hat{\mathbf{s}}_1 \\ \hat{\mathbf{s}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \\ \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{bmatrix} \quad (9)$$

Now, let us consider circuit, which consists of any N subcircuits. This case can be described by general matrix equation:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{s}}} \end{bmatrix} + \begin{bmatrix} \mathbf{A} & \mathbf{G} \\ \mathbf{B} & \mathbf{N} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{s}} \end{bmatrix} = \begin{bmatrix} \mathbf{D} \\ \mathbf{C} \end{bmatrix} \mathbf{s} \quad (10)$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \cdots \\ \mathbf{x}_N \end{bmatrix}_{M_1 \times 1} \quad \hat{\mathbf{s}} = \begin{bmatrix} \hat{\mathbf{s}}_1 \\ \cdots \\ \hat{\mathbf{s}}_N \end{bmatrix}_{M_2 \times 1} \quad \mathbf{s} = \begin{bmatrix} \mathbf{s}_1 \\ \cdots \\ \mathbf{s}_N \end{bmatrix}_{M_3 \times 1} \quad (11)$$

moreover, $\mathbf{x}_n \in \mathbf{R}^{N_n}$, $\mathbf{s}_n \in \mathbf{R}^{Z_n}$, $\hat{\mathbf{s}}_n \in \mathbf{R}^{\hat{Z}_n}$, $\mathbf{M} = \mathrm{diag}(\mathbf{M}_1, \ldots, \mathbf{M}_N) \in \mathbf{R}^{M_1 \times M_1}$ a $\mathbf{M}_n \in \mathbf{R}^{N_n \times N_n}$ $(n = 1 \ldots N)$, $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_N) \in \mathbf{R}^{M_1 \times M_1}$ a $\mathbf{A}_n \in \mathbf{R}^{N_n \times N_n}$ $(n = 1 \ldots N)$, $\mathbf{G} = \mathrm{diag}(\mathbf{G}_1, \ldots, \mathbf{G}_N) \in \mathbf{R}^{M_1 \times M_2}$ a $\mathbf{D} = \mathrm{diag}(\mathbf{D}_1, \ldots, \mathbf{D}_N) \in \mathbf{R}^{M_1 \times M_3})$, $\mathbf{B} \in \mathbf{R}^{M_2 \times M_1}$, $\mathbf{C} \in \mathbf{R}^{M_2 \times M_3}$, $\mathbf{N} = \mathrm{diag}(1, \ldots, 1) \in \mathbf{R}^{M_2 \times M_2}$.

A type of the chosen WR procedures results in a type of splittings of the structural matrices into two parts: $\mathbf{M} = \mathbf{M}_a - \mathbf{M}_b$, $\mathbf{A} = \mathbf{A}_a - \mathbf{A}_b$, $\mathbf{B} = \mathbf{B}_a - \mathbf{B}_b$, $\mathbf{C} = \mathbf{C}_a - \mathbf{C}_b$, $\mathbf{D} = \mathbf{D}_a - \mathbf{D}_b$, $\mathbf{G} = \mathbf{G}_a - \mathbf{G}_b$, $\mathbf{N} = \mathbf{N}_a - \mathbf{N}_b$.

$$\begin{bmatrix} \mathbf{M}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{s}}} \end{bmatrix}^{(j+1)} + \begin{bmatrix} \mathbf{A}_a & \mathbf{G}_a \\ \mathbf{B}_a & \mathbf{N}_a \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{s}} \end{bmatrix}^{(j+1)} = \begin{bmatrix} \mathbf{M}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{s}}} \end{bmatrix}^{(j)} + \begin{bmatrix} \mathbf{A}_b & \mathbf{G}_b \\ \mathbf{B}_b & \mathbf{N}_b \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{s}} \end{bmatrix}^{(j)} + \begin{bmatrix} \mathbf{D} \\ \mathbf{C} \end{bmatrix} \mathbf{s} \quad (12)$$

For the block Jacobi WR method we take as follows: $\mathbf{M}_b = \mathbf{0}$ i $\mathbf{A}_b = \mathbf{0}$, $\mathbf{M}_b = \mathbf{0}$, $\mathbf{G}_a = \mathbf{0}$, $\mathbf{B}_b = \mathbf{0}$, $\mathbf{N}_b = \mathbf{0}$.

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{s}}} \end{bmatrix}^{(j+1)} + \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{N} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{s}} \end{bmatrix}^{(j+1)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{s}}} \end{bmatrix}^{(j)} + \begin{bmatrix} \mathbf{0} & \mathbf{G} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{s}} \end{bmatrix}^{(j)} + \begin{bmatrix} \mathbf{D} \\ \mathbf{C} \end{bmatrix} \mathbf{s} \quad (13)$$

B. Multisplittings

Let L denote the number of different splittings arisen from the process of circuit decomposition.

Splitting denoted by k is defined as well by the structural matrices decompositions as by the additional weight matrix $\mathbf{E}\big|_k \in \mathbf{R}^{(M_1+M_2)\times(M_1+M_2)}$.

$$\begin{bmatrix}\mathbf{M}_{a,k} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\cdot\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\widehat{\mathbf{s}}}\end{bmatrix}_k^{(j+1)}+\begin{bmatrix}\mathbf{A}_{a,k} & \mathbf{G}_{a,k}\\ \mathbf{B}_{a,k} & \mathbf{N}_{a,k}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{x}\\ \widehat{\mathbf{s}}\end{bmatrix}_k^{(j+1)}=$$
$$=\begin{bmatrix}\mathbf{M}_{b,k} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\cdot\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\widehat{\mathbf{s}}}\end{bmatrix}^{(j)}+$$
$$+\begin{bmatrix}\mathbf{A}_{b,k} & \mathbf{G}_{b,k}\\ \mathbf{B}_{b,k} & \mathbf{N}_{b,k}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{x}\\ \widehat{\mathbf{s}}\end{bmatrix}^{(j)}+\begin{bmatrix}\mathbf{D}\\ \mathbf{C}\end{bmatrix}\mathbf{s} \quad (14)$$

The final result of the j+1 iteration is

$$\begin{bmatrix}\mathbf{x}\\ \widehat{\mathbf{s}}\end{bmatrix}^{(j+1)}=\sum_{k=1}^{k=L}\mathbf{E}\Big|_k\begin{bmatrix}\mathbf{x}\\ \widehat{\mathbf{s}}\end{bmatrix}_k^{(j+1)} \quad (15)$$

and the relation $\mathbf{E}\big|_1+\mathbf{E}\big|_2+\cdots+s\,\mathbf{E}\big|_L=\mathrm{diag}(1,\ldots,1)$ must be fulfilled.

Let us describe Jacobi WR method used with the concept of multisplittings. We assume that $L=N$ and that the splitting of circuit matrices doesn't vary at every stage k ($k=1,\ldots,N$). It means, the splitting description given by (13) stays the same. We will change weight matrices $\mathbf{E}\big|_k$ in such a manner, that, at every stage k, we need to know only the values of vector variables $\mathbf{x}_k$, $\widehat{\mathbf{s}}_k$, which can be calculated directly from the subcircuit k.

Let us explain the weight matrix construction. For that purpose we introduce the notations: $\mathbf{E}^{(\mathbf{x})}\big|_k \in \mathbf{R}^{M_1\times M_1}$ i $\mathbf{E}^{(\widehat{\mathbf{s}})}\big|_k \in \mathbf{R}^{M_2\times M_2}$.

$$\mathbf{E}\Big|_k=\begin{bmatrix}\mathbf{E}^{(\mathbf{x})}\big|_k & \mathbf{0}\\ \mathbf{0} & \mathbf{E}^{(\widehat{\mathbf{s}})}\big|_k\end{bmatrix} \quad (16)$$

For $k=1,\ldots,N$ we set zeros into matrices $\mathbf{E}^{(\mathbf{x})}\big|_k$ and $\mathbf{E}^{(\widehat{\mathbf{s}})}\big|_k$, except the diagonal elements corresponding to the $\mathbf{x}_k$ and $\widehat{\mathbf{s}}_k$ variables respectively, where we set 1.

C. Subcircuits overlapping

The basic idea of overlapping techniques is realized here in such a manner, that two subcircuits will be analyzed simultaneously and the weight matrices are of the form:

$$\mathbf{E}\Big|_k=\alpha\begin{bmatrix}\mathbf{E}^{(\mathbf{x})}\big|_m & \mathbf{0}\\ \mathbf{0} & \mathbf{E}^{(\widehat{\mathbf{s}})}\big|_m\end{bmatrix}+$$
$$+(1-\alpha)\begin{bmatrix}\mathbf{E}^{(\mathbf{x})}\big|_n & \mathbf{0}\\ \mathbf{0} & \mathbf{E}^{(\widehat{\mathbf{s}})}\big|_n\end{bmatrix} \quad (17)$$

where $\alpha \in <0,1>$ and for $k=1,2,\ldots,N$ pairs of numbers m,n, have the following values: (1,2) (2,3) ... (N,1).

D. Convergence condicions

Convergence condition for the algorithm described by formula (14) is of the form [13]:

$$\rho\left(\sum_{k=1}^{L}\mathbf{E}^{(\mathbf{x})}\Big|_k\,\mathbf{M}_{a,k}^{-1}\,\mathbf{M}_{b,k}\right)<1\,, \quad (18)$$

$$\rho\left(\sum_{k=1}^{L}\mathbf{E}^{(\mathbf{x})}\Big|_k\,\mathbf{N}_{a,k}^{-1}\,\mathbf{N}_{b,k}\right)<1\,, \quad (19)$$

where $\rho(*)$ is an operator of finding spectral radius of a given matrix. Because $\mathbf{M}_{b,k}=\mathbf{0}$ and $\mathbf{N}_{b,k}=\mathbf{0}$ hold for $k=1,\ldots,N+1$, convergence of the investigated algorithm is always ensured.

E. Nonlinear circuit analysis

Let us consider description of the splitted system:

$$\begin{aligned}\mathbf{f}_{\mathbf{x}}(\dot{\mathbf{x}},\mathbf{x},\widehat{\mathbf{s}},\mathbf{s},t)=\mathbf{0}\,, \qquad \mathbf{x}(0)=\bar{\mathbf{x}}\\ \mathbf{f}_{\widehat{\mathbf{s}}}(\mathbf{x},\widehat{\mathbf{s}},\mathbf{s},t)=\mathbf{0}\,,\end{aligned} \quad (20)$$

where $\mathbf{f}_{\mathbf{x}}=[\mathbf{f}_{\mathbf{x},1}\ldots\mathbf{f}_{\mathbf{x},N}]^T$, $\mathbf{f}_{\widehat{\mathbf{s}}}=[\mathbf{f}_{\widehat{\mathbf{s}},1}\ldots\mathbf{f}_{\widehat{\mathbf{s}},N}]^T$ and $\mathbf{f}_{\mathbf{x},n}\in\mathbf{R}^{M_{1,n}}$, $\mathbf{f}_{\widehat{\mathbf{s}},n}\in\mathbf{R}^{M_{2,n}}$, wheras $n=1,\ldots,N$, $M_{1,1}+\cdots+M_{1,N}=M_1$, $M_{2,1}+\cdots+M_{2,N}=M_2$. We are looking for a solution vector $\mathbf{x}(t)$, which satisfies equations (20) over a given finite interval $[0,T]$.

Thus, the equations describing subcircuit number n are of the form:

$$\begin{aligned}\mathbf{f}_{\mathbf{x},n}\left(\dot{\mathbf{x}}_n,\mathbf{x}_n,\widehat{\widehat{\mathbf{s}}}_n,\mathbf{s}_n,t\right)=\mathbf{0}\,, \qquad \mathbf{x}_n(0)=\bar{\mathbf{x}}_n\\ \mathbf{f}_{\widehat{\mathbf{s}},n}(\mathbf{x}_n,\widehat{\mathbf{s}}_n,\mathbf{s}_n,t)=\mathbf{0}\,,\end{aligned} \quad (21)$$

where $\widehat{\widehat{\mathbf{s}}}_n$ is a vector containing sources acting inside subcircuit n, and $\widehat{\mathbf{s}}_n$ contains sources supplying other subcircuits, but calculated in relation with $\mathbf{x}_n$.

F. Algorithm

1. **Iterate:** For $j=1,2,\ldots$ to satisfied do:
2. **Set:** $\mathbf{x}^{(j)}(0)=\bar{\mathbf{x}}$.
3. **Solve:** for $k=1,\ldots,N$

$$\begin{cases}\mathbf{f}_{\mathbf{x},n}\left(\dot{\mathbf{x}}_n^{(j+1)}\big|_k,\mathbf{x}_n^{(j+1)}\big|_k,\widehat{\widehat{\mathbf{s}}}_n^{(j)},\mathbf{s}_n,t\right)=\mathbf{0}\,,\\ \mathbf{f}_{\widehat{\mathbf{s}},n}\left(\mathbf{x}_n^{(j+1)}\big|_k,\widehat{\mathbf{s}}_n^{\,j+1}\big|_k,\mathbf{s}_n,t\right)=\mathbf{0}\\ \mathbf{f}_{\mathbf{x},n+1}\left(\dot{\mathbf{x}}_{n+1}^{(j+1)}\big|_k,\mathbf{x}_{n+1}^{(j+1)}\big|_k,\widehat{\widehat{\mathbf{s}}}_{n+1}^{(j)},\mathbf{s}_{n+1},t\right)=\mathbf{0}\,,\\ \mathbf{f}_{\widehat{\mathbf{s}},n+1}\left(\mathbf{x}_{n+1}^{(j+1)}\big|_k,\widehat{\mathbf{s}}_{n+1}^{\,j+1},\mathbf{s}_{n+1},t\right)=\mathbf{0}\end{cases} \quad (22)$$

to obtain $\mathbf{x}^{(j+1)}\big|_k$ and $\widehat{\mathbf{s}}^{\,j+1}\big|_k$.

4. **Calculate:** $\mathbf{x}^{(j+1)}$ by the relationship

$$\begin{bmatrix}\mathbf{x}\\ \widehat{\mathbf{s}}\end{bmatrix}^{(j+1)}=\sum_{k=1}^{N}\mathbf{E}\Big|_k\begin{bmatrix}\mathbf{x}\\ \widehat{\mathbf{s}}\end{bmatrix}_k^{(j+1)} \quad (23)$$

where $\mathbf{E}\big|_k$ is expressed by equation (17).

To study convergence of the presented algorithm we apply Newton's method in functional space [16]. As a result we approximate nonlinear equations (20) by the linear system with time-varying coefficients

$$\begin{bmatrix}\mathbf{M}_a(t) & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\cdot\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\hat{\mathbf{s}}}\end{bmatrix}^{(j+1)}+\begin{bmatrix}\mathbf{A}_a(t) & \mathbf{G}_a(t)\\ \mathbf{B}_a(t) & \mathbf{N}_a(t)\end{bmatrix}\cdot\begin{bmatrix}\mathbf{x}\\ \hat{\mathbf{s}}\end{bmatrix}^{(j+1)}=$$
$$=\begin{bmatrix}\mathbf{M}_b(t) & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\cdot\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\hat{\mathbf{s}}}\end{bmatrix}^{(j)}+\begin{bmatrix}\mathbf{A}_b(t) & \mathbf{G}_b(t)\\ \mathbf{B}_b(t) & \mathbf{N}_b(t)\end{bmatrix}\cdot\begin{bmatrix}\mathbf{x}\\ \hat{\mathbf{s}}\end{bmatrix}^{(j)}+$$
$$+\begin{bmatrix}\mathbf{D}(t)\\ \mathbf{C}(t)\end{bmatrix}\mathbf{s}\,, \quad (24)$$

where $\mathbf{M}(t)=\mathbf{M}_a(t)-\mathbf{M}_b(t)$ and $\mathbf{N}(t)=\mathbf{N}_a(t)-\mathbf{N}_b(t)$ are nonsingular matrices for all $t\in[0,T]$. Waveform method is convergent [2] if

$$\max_{t\in[0,T]}\rho\left(\mathbf{M}_a^{-1}(t)\,\mathbf{M}_b(t)\right)<1 \qquad \text{and}$$
$$\max_{t\in[0,T]}\rho\left(\mathbf{N}_a^{-1}(t)\,\mathbf{N}_b(t)\right)<1\,. \quad (25)$$

We assume $\mathbf{M}_b(t)=\mathbf{0}$ and $\mathbf{N}_b(t)$ for every $t\in[0,T]$, hence the condition (25) is fulfilled for the given iteration procedure.

III. Numerical example

To investigate efficiency of the proposed algorithm we have analysed a simple testing circuit (the N-segment model of transmission line with nonlinear conductance) presented in Fig.4. We compared standard block Jacobi WR algorithm with the proposed multisplitting procedure for the wide range of overlapping coefficients. Calculating two neighbouring segments and using the factor $\alpha=0.6$ we have speeded up the iteration process by 30% (average result for different values of model elements). We have used zero initial vector, error tolerance $\varepsilon=10^{-4}$ and trapezoidal rule as basic code of ODEs. As a stopping criteria we require

$$\left\|\mathbf{x}^{(j+1)}(t)-\mathbf{x}^{(j)}(t)\right\|\le\varepsilon\,,$$

for all $t\in[0,T]$.

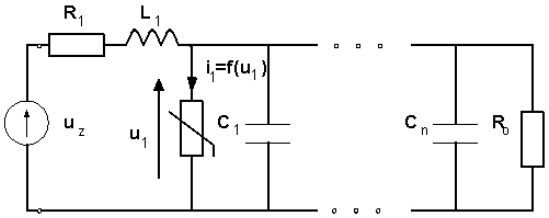

Fig. 4. Transmission line model for numerical test

IV. Conclusion

The concept of waveform relaxation algorithm presented above is based on the special multisplitting of the nonlinear circuits. Partial results are relaxed with overlapping techniques. Applying this conncept we can ensure numerical stability of the WR iterations performed for the nonlinear circuit having nonsingular structural matrices fulfilling the convergence criteria (25). Hence, there is no need to check convergence of the procedure in every step of the discrete numerical iteration. The numerical experiments showed that applying proposed method we can improve the efficiency of the WR algorithm by more than 30%. The proper choice of decomposition and optimisation of the linking structure may speed up the method significantly, but the general rules are very hard to be uniquely defined. The use of different overlapping coefficient a for different subcircuits is still investigated by authors and today results looks promising.

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