3D Electrical Impedance Tomography forward problem solution approximated by Boundary Element Method

Magdalena Stasiak, Jan Sikora, Stefan F. Filipowicz

Abstract — **In Electrical Impedance Tomography (EIT) as a default 3D diagnostic method, the most challenging task is to determine the distribution of potential in specific organs. In order to simulate and collect values of the potential, the adequate model of computer simulation and the forward problem solution methods are needed. The Finite Element Method (FEM) and also the Boundary Element Method (BEM) are the most frequently used among many others. The Finite Elements Method, which is time consuming in a 3D space, is usually used to the solution of the forward problem. In the paper the BEM, which represents only a discretization of the surface and reduces the number of necessary element (as a consequence the computation time) is presented.**

I. Introduction

THE Boundary Element Method (BEM) is a numerical technique for calculating the surface potential generated by current sources located in piecewise homogenous volume conductor [4], [5], [10], [12]. This method is capable of providing a solution to a volume problem by calculating the effects of the source at the boundaries of the volume. The boundaries are the interfaces between regions of different conductivity within the volume and also the outer surface [4], [5].

First step of the BEM is mesh generation. Meshing can be defined as the process of division a physical domain into smaller sub-domains (elements) in order to facilitate the numerical solution of a partial differential equation. In BEM the partial differential equations with adequate boundary conditions are transformed into equivalent integral equations set defined only on the surface of the considered volume.

In the numerical realization of BEM the form of boundary elements depend on the problem dimension. For the 1D problem – the boundary elements are reduced to points, for 2D problem are the form of rectilinear or curved segments, and for 3D problem the boundary elements are triangular or quadrangular parts of a surface [1], [2].

The original integral equation governing surface potential can be approximated as a summation of surface integrals of each element. While the potential on each element remains to be calculated, it may be assumed to be a simple basis function of some unknowns. The potential on an element may be modeled as [1], [2], [5], [7], [12]:

- a constant value (zero-order interpolation functions) – the function and partial derivative are constant on each element,
- a linear functions, which attains a different value at each of the vertices of the element and the function and partial derivative are linear,
- a square function – the function and partial derivative are described as a quadric function of the local variables.

The finite-dimensional approximation of boundary integral equations consequently reduces them to algebraic equations set. Its solution allows determining all unknown function on the boundary of the volume. In the next step of BEM finding the unknown function in the inside points is possible, without necessity of discretization of this domain [2], [7].

II. Factors affecting the accuracy of BEM in the forward problem

The factors, which limit the accuracy and affect the time required to obtain a solution using the BEM are:

- the choice of basis function,
- density of elements,
- shape of each element.

The choice of basis functions typically is limited to either constant or linear basis functions on triangular elements.

Element density depends on the shape and size of elements, limiting the extent and shape of actual surface. A higher element density provides a more accurate representation of the shape and a more precise solution [5], [7], [12]. But the total computational time of forward problem solution dramatically increases for more number of elements. The expense of this solution is unacceptable, while the main requirements for EIT application are focused on image reconstruction in the real time.

Element shape is most often restricted to flat triangular elements to model curved body surfaces, due to the underlying simplicity in the mathematical formulation of the problem [5], [12].

Regardless of the number of considered and modelled surfaces, the BEM using for realistic-shaped surfaces is connected with selection of available basis functions and using a specific grid pattern which determines element density and shape.

Magda Stasiak is with the Institute of Electrical Apparatus, Technical University in Lodz, ul. Stefanowskiego 18/22, 90-924 Lodz, Poland, e-mail: *stasiak@p.lodz.pl.*

Jan Sikora and Stefan Filipowicz are with the Institute of Theory of Electrical Engineering, Measurement and Information Systems, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warszawa, Poland, e-mail: *sik{2xf}@iem.pw.edu.pl*

III. BEM FOR LAPLACE'S EQUATION

Integral equations are used to solution to the forward problems (analisys) and the inverse problems (synthesis and identification) [13]. To the transformation of the integral equations, the function and Green's formulas are used. For the 3D domain, the Green's function is:

$$G'(A,B) = \frac{1}{4\pi r} + g(A,B) \tag{1}$$

gdzie: $G'(A,B) = G(A,B) + g(A,B)$
and

$G(A, B)$ – Green's function for boundless domain
$g(A,B)$ – function responsible for satisfying by $G'(A, B)$ boundary conditions

The second Green's identity (symmetrical), which was used in transformation of the integral equation, is described as follows:

$$\int_{\Omega} \left(\Phi \nabla^2 G - G \nabla^2 \Phi\right) d\Omega = \oint_{S} \left(\Phi \frac{\partial G}{\partial n} - G \frac{\partial \Phi}{\partial n} \right) dS \tag{2}$$

gdzie: Φ - potential in 3D domain Ω

Let consider the Lapace's equation in the three-dimensional domain:

$$\nabla^2 \Phi(r) = 0 \ \forall_{r \in \Omega} \tag{3}$$

for the Dirichlet boundary conditions on the surface of the volume.

The Green's function satisfies the Laplace's equation:

$$\nabla^2 G = 0 \tag{4}$$

and for 3D domain is:

$$G(M,P) = \frac{1}{4\pi r} \tag{5}$$

where:
r – the distance between M and P

$$r(M,P) = \sqrt{(x_M - x_P)^2 + (y_M - y_P)^2 + (z_M - z_P)^2} \tag{6}$$

Using equation (2) – second Green's identity we get:

$$\Phi(M) + \int_{\Gamma} K_1(M,P) \Phi(P) d\Gamma(P) = \int_{\Gamma} K_2(M,P) \frac{\partial \Phi}{\partial n} d\Gamma(P) \tag{7}$$

where functions K_1 and K_2 are kernels of integral transformation, which are defined as follows [10], [12]:

$$K_1(M,P) = \frac{\partial G(M,P)}{\partial n} \tag{8}$$

$$K_2(M,P) = G(M,P) \tag{9}$$

Let move the interior load point M to the boundary which results in the following equation:

$$C(P_0)\Phi(P_0) + \int_{\Gamma} K_1(P_0,P)\Phi(P) d\Gamma(P) = \\ = \int_{\Gamma} K_2(P_0,P) \frac{\partial \Phi(P)}{\partial n} d\Gamma(P) \tag{10}$$

The coefficients in equation (10) are identified with inside point of Ω domain or boundary point $C(P_0)$. In the case, when the point P is located in the within Ω domain, $C(P_0)=1$, and $C(P_0)=0.5$ for boundary points, on the assumption that boundary is smooth. In the differe*n*t case this coefficient depends on the solid angle (for 3D) in the point of refraction [2], [10]:

$$C_i = \frac{\Theta_i}{2(n-1)\pi} \tag{11}$$

where: n – dimension of space
Θ - solid angle (for n=3)

To calculate the first kernel $K_1(M,P)$, the Green's function is differentiated with respect to unit normal at the point P, as follows [10], [12]:

$$\frac{\partial G(M,P)}{\partial n} = \frac{\partial G}{\partial r}\left(\frac{\partial r}{\partial n}\right) = \frac{\partial G}{\partial r}\left[\frac{\partial r}{\partial x}\left(\frac{\partial x}{\partial n}\right) + \frac{\partial r}{\partial y}\left(\frac{\partial y}{\partial n}\right) + \frac{\partial r}{\partial z}\left(\frac{\partial z}{\partial n}\right)\right] \tag{12}$$

where the derivatives of the coordinates x, y, z with respect to the unit outward normal n in point P are the components of the outward normal as follows:

$$n_x = \frac{\partial x}{\partial n},\ n_y = \frac{\partial y}{\partial n},\ n_z = \frac{\partial z}{\partial n} \tag{13}$$

and

$$\frac{\partial r(M,P)}{\partial x} = \frac{x_P - x_M}{r(M,P)} \\ \frac{\partial r(M,P)}{\partial y} = \frac{y_P - y_M}{r(M,P)} \\ \frac{\partial r(M,P)}{\partial z} = \frac{z_P - z_M}{r(M,P)} \tag{14}$$

Therefore, the first kernel can be written:

$$K_1(M,P) = \frac{1}{4\pi r^3}\left[(x_P - x_M)n_x + (y_P - y_M)n_y + (z_P - z_M)n_z\right] \tag{15}$$

To solve the three-dimensional problem numerically, the surface has to be discretized into elements. The element, which was modeled as a constant value have been used in this case.

IV. ZERO-ORDER INTERPOLATION FUNCTIONS

Let consider the zero-order interpolation functions for flat triangular element [1], [2], [7], [10], [12].

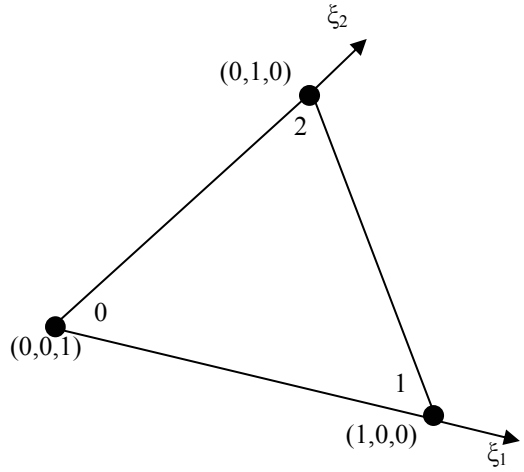

Fig. 1.Local coordinates of the triangle

$$\begin{aligned} N_0(\xi_1,\xi_2) &= 1-\xi_1-\xi_2 \\ N_1(\xi_1,\xi_2) &= \xi_1 \\ N_2(\xi_1,\xi_2) &= \xi_2 \end{aligned} \tag{16}$$

The first derivatives of the standard interpolation functions with respect to the ξ_1 and ξ_2 are given by:

$$\begin{aligned} \frac{\partial N_0(\xi_1,\xi_2)}{\partial \xi_1} &= -1 \\ \frac{\partial N_1(\xi_1,\xi_2)}{\partial \xi_1} &= 1 \\ \frac{\partial N_2(\xi_1,\xi_2)}{\partial \xi_1} &= 0 \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{\partial N_0(\xi_1,\xi_2)}{\partial \xi_2} &= -1 \\ \frac{\partial N_1(\xi_1,\xi_2)}{\partial \xi_2} &= 0 \\ \frac{\partial N_2(\xi_1,\xi_2)}{\partial \xi_2} &= 1 \end{aligned} \tag{18}$$

The singular integrals in zero-order triangular element over a flat area can be evaluated analytically [12]. The symbols used in the following closed forms are shown in Fig. 2.

$$\begin{aligned} \int_\Delta \frac{1}{r} dS = \frac{2\Delta}{3}\Bigg[& \frac{1}{r_{12}} \ln\left|\frac{\tan[(\Theta_0+\alpha_1)/2]}{\tan(\alpha_1/2)}\right| + \\ & + \frac{1}{r_{20}} \ln\left|\frac{\tan[(\Theta_1+\alpha_2)/2]}{\tan(\alpha_2/2)}\right| + \\ & + \frac{1}{r_{01}} \ln\left|\frac{\tan[(\Theta_2+\alpha_0)/2]}{\tan(\alpha_0/2)}\right| \Bigg] \end{aligned} \tag{19}$$

$$\int_\Delta \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS = 0 \tag{20}$$

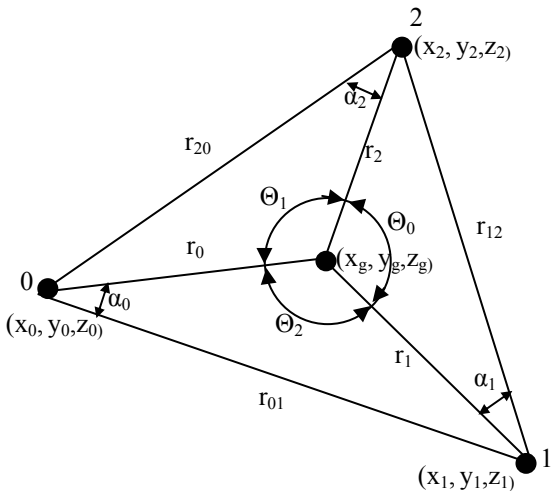

Fig. 2. Notations for geometric information of triangular element

V. THE BOUNDARY CONDITIONS

For Dirichlet boundary conditions vector $\Phi = \Phi_D$ is specified, and vector $\frac{\partial \Phi}{\partial n}$ is unknown. For BEM these boundary conditions could be formulated as follows:

$$B\frac{\partial \Phi}{\partial n} = A\Phi_D \tag{21}$$

In the Neumann boundary conditions, the vector $\frac{\partial \Phi}{\partial n} = \left(\frac{\partial \Phi}{\partial n}\right)_N$ is known, and vector Φ is unknown.

$$A\Phi = \left(B\frac{\partial \Phi}{\partial n}\right)_N \tag{22}$$

In the case of the solution to the forward problem in EIT, only *2* elements are imposed Dirichlet boundary conditions, and the other *m-2* the Neumann boundary conditions.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Phi_D \\ \Phi \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \Phi}{\partial n} \\ \left(\dfrac{\partial \Phi}{\partial n}\right)_N \end{bmatrix} \tag{23}$$

After transformation we get:

$$\begin{bmatrix} -B_{11} & A_{12} \\ -B_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \Phi}{\partial n} \\ \Phi \end{bmatrix} = \begin{bmatrix} B_{12}\left(\dfrac{\partial \Phi}{\partial n}\right)_N - A_{11}\Phi_D \\ B_{22}\left(\dfrac{\partial \Phi}{\partial n}\right)_N - A_{21}\Phi_D \end{bmatrix} \tag{24}$$

For $\left(\frac{\partial \Phi}{\partial n}\right)_N = 0$ the equation is formulated as follows:

$$\begin{bmatrix} -B_{11} & A_{12} \\ -B_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \Phi}{\partial n} \\ \Phi \end{bmatrix} = \begin{bmatrix} -A_{11}\Phi_D \\ -A_{21}\Phi_D \end{bmatrix} \tag{25}$$

VI. THE FORWARD PROBLEM IN EIT

The suggested model consists of three layers with

adequate number of electrodes: 16, 8 and 4 (Fig.3). Some instances of electrodes arrangement and the way of electrode-to-electrode collection have been given in [6] and [14]. The 3D breast phantom for Optical Tomography with three layers of optodes and 3D images produced using data acquired by 29 or 30 detectors per source (for 32 source position) is described by [6]. The different approach is presented in [14], where the measurement chamber is a hemisphere of 16-cm diameter with 64 compound electrodes placed in fixed position.

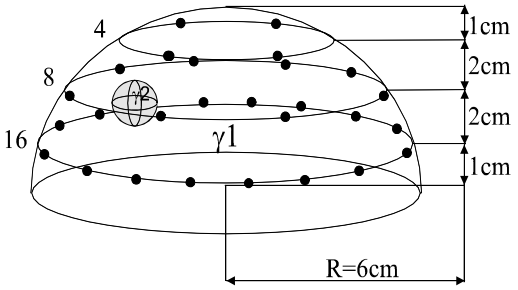

Fig.3. Model of computer simulation and arrangement of electrodes

In order to generate mesh we use Netgen [8], [9]. Simulations have been conducted for sphere-shaped inside object with radius R=0.8, 1.0, 1.1, 1.2, 1.4, 1.5, 1.6, 1.8, 2.0, 2.2, 2.4, 2.5 cm and for its different locations.

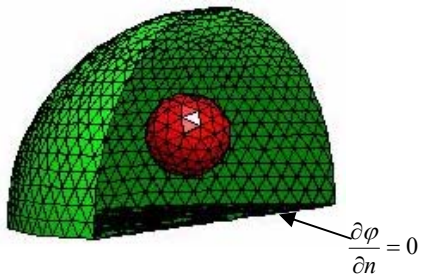

Fig. 4. Generated surface mesh

VII. Conclusion

The main advantages of using BEM in EIT are:

- dyscretization only boundary of analysed domain, what is connected with reduction the dimension of the problem and total computational time,
- finding the unknown function in the inside points is possible, without necessity of discretization of this domain,
- this method allows to analyse boundless domain,
- possibility of approach the electromagnetic field analysis.

ACKNOWLEDGEMENTS

This work was supported by Polish Science Research Committee (grant in 2002-2004).

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