An Algorithm for the Analysis of Dynamic Electronic Circuits

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Abstract — **The paper deals with nonlinear dynamic circuits and brings an efficient algorithm for the transient analysis of a broad class of electronic circuits. An earlier developed numerical-integration method has been implemented using associated discrete models of capacitors and inductors. The algorithm overcomes drawbacks of the well known and commonly used trapezoidal and Gear's methods and usually is less time consuming. Two numerical examples, given in the paper, confirm advantages of the algorithm.**

I. Introduction

Transient analysis of nonlinear dynamic circuits is a basic question of the design of electronic circuits. If a circuit is described by a state equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{1}$$

then the solution $\boldsymbol{x}(t)$ can be found using a multistep-integration method [1]-[8]. The trapezoidal scheme is assumed to be the best of general-purpose methods for solving the initial-value problem (1), whereas Gear's scheme is the most useful for solving stiff state space equations. These methods are implemented in SPICE. Although the trapezoidal and Gear methods have very good stability properties they suffer from the following shortcomings. The trapezoidal method may generate spurious oscillations when the transient solutions change fast. On the other hand Gear's method may give damped oscillations whereas the real solutions are sustained or unstable. To overcome these drawbacks the step size should be decreased to very small value, when the transient solutions vary very fast. However, if the step size is too small, a large number of time steps will be necessary to cover the specified solution time interval and the amount of computation will increase substantially. Moreover, the step size cannot be too small if we wish to obtain the solution in the real time. This is why we assume in this paper a limit for the step size.

To avoid drawbacks of the above mentioned methods a new family of numerical integration methods has been developed in [9]. All the methods are implicit, second-order, *A*-stable and they depend on a parameter which is allowed to be changed during the computation process according to a proposed strategy. The variable step-size version of the method is as follows

$$x_{k+2} = -\theta \frac{h_2^2}{h_1(h_1+2h_2)} x_k + \left(1+\theta \frac{h_2^2}{h_1(h_1+2h_2)}\right) x_{k+1} + (1-\theta)\frac{h_2}{2}\dot{x}_{k+1} + \left(1+\theta\frac{h_1}{h_1+2h_2}\right)\frac{h_2}{2}\dot{x}_{k+2} \tag{2}$$

where $\theta \in [0,1]$. In the special case where $h_1 = h_2 = h$, equation (2) reduces to

$$x_{k+2} = -\frac{1}{3}\theta x_k + \left(1+\frac{1}{3}\theta\right)x_{k+1} + (1-\theta)\frac{h_2}{2}\dot{x}_{k+1} + \left(1+\frac{1}{3}\theta\right)\frac{h_2}{2}\dot{x}_{k+2}\,. \tag{3}$$

For the end values $\theta = 0$ and $\theta = 1$ (3) becomes the trapezoidal scheme [5]

$$x_{k+2} = x_{k+1} + \frac{h}{2}(\dot{x}_{k+1} + \dot{x}_{k+2}) \tag{4}$$

or the Gear scheme [5]

$$x_{k+2} = -\frac{1}{3}x_k + \frac{4}{3}x_{k+1} + \frac{2}{3}h\,\dot{x}_{k+2}\,. \tag{5}$$

A general-purpose method for solving the initial-value problem must allow the step size to be varied. The method should change to larger step size when the transient has changed slowly and to smaller step size when the transient has changed fast. This is why we use the variable step size version (2). It can be shown that the principal local truncation error of the method (2) is given by

$$\varepsilon_T = -\left(\frac{1}{12}h_2^3 + \theta\frac{h_1 h_2^2(2h_1+3h_2)}{12(h_1+2h_2)}\right)\dddot{x}(\tau)\,. \tag{6}$$

The computation process is carried out using the following strategy. We compute $\boldsymbol{\varepsilon}_T = [(\varepsilon_1)_T \ldots (\varepsilon_n)_T]^{\mathrm{T}}$ and divide each component of this vector by the corresponding component of $\boldsymbol{x}_{k+2}$. Next we find the quantity having the largest absolute value and label it M. If $10^{-4} \le M \le 10^{-3}$ the step remains unchanged. If $M > 10^{-3}$ the result obtained is skipped and the step is decreased twice. Simultaneously the parameter θ is increased according to the formula: $\theta_{new} = 0.2 + 0.8\theta$. If $M < 10^{-4}$ in two subsequent points the step is doubled and θ is decreased according to the formula $\theta_{new} = 0.8\theta$. The above procedure is realized for the

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step sizes framed by the assumed minimal and maximal values.

II. Efficient implementation of the method

Many nonlinear dynamic circuits encountered in practice do not have the state representation (1) and the method described in Section I cannot be directly used. Therefore, in order to implement the method we apply the known idea of discrete models of capacitors and inductors.

Let us consider a nonlinear capacitor, shown in Fig. 1, described by

$$q = \hat{q}(v), \qquad i = \frac{\mathrm{d}q}{\mathrm{d}t} \tag{7}$$

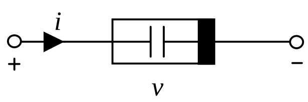

Fig. 1. A nonlinear capacitor.

and apply equation (2) rearranged to the form

$$\begin{aligned} \dot{x}_{k+2} &= a_1(h_1, h_2, \theta)x_{k+2} + a_2(h_1, h_2, \theta)x_{k+1} + \\ &+ a_3(h_1, h_2, \theta)x_k + b(h_1, h_2, \theta)\dot{x}_{k+1} \end{aligned} \tag{8}$$

where

$$a_1(h_1, h_2, \theta) = \frac{2(h_1 + 2h_2)}{((1+\theta)\, h_1 + 2h_2)h_2},$$

$$a_2(h_1, h_2, \theta) = -\frac{2\left(h_1(h_1 + 2h_2) + \theta h_2^2\right)}{h_1 h_2 (h_1 + 2h_2) + \theta h_1^2 h_2},$$

$$a_3(h_1, h_2, \theta) = \frac{2\theta h_2}{(1+\theta)h_1^2 + 2h_1 h_2},$$

$$b = -\frac{(1-\theta)(h_1 + 2h_2)}{(1+\theta)h_1 + 2h_2}.$$

As a result we obtain

$$i_{k+2} = g(v_{k+2}) + I_{k+2} \tag{9}$$

where

$$g(v_{k+2}) = a_1(h_1, h_2, \theta)\hat{q}(v_{k+2}) \tag{10}$$

$$\begin{aligned} I_{k+2} &= a_2(h_1, h_2, \theta)\hat{q}(v_{k+1}) + a_3(h_1, h_2, \theta)\hat{q}(v_k) + \\ &+ b(h_1, h_2, \theta)i_{k+1}. \end{aligned} \tag{11}$$

Equation (9) describes a discrete model of the nonlinear capacitor composed of a nonlinear resistor and current source connected in parallel (see Fig. 2), where the source current depends on the previously computed voltages v_k, v_{k+1} and current i_{k+1}.

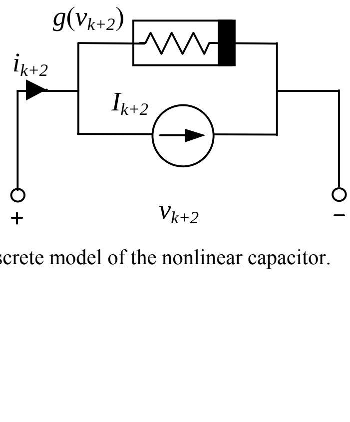

Fig. 2. Discrete model of the nonlinear capacitor.

If the capacitor is linear $(q = Cv)$ then equation (9) becomes the linear equation

$$i_{k+2} = Gv_{k+2} + \hat{I}_{k+2} \tag{12}$$

$$G = a_1(h_1, h_2, \theta)C \tag{13}$$

$$\begin{aligned} \hat{I}_{k+2} &= a_2(h_1, h_2, \theta)Cv_{k+1} + a_3(h_1, h_2, \theta)Cv_k + \\ &+ b(h_1, h_2, \theta)i_{k+1}. \end{aligned} \tag{14}$$

Equation (12) represents a discrete model of the linear capacitor, as shown in Fig. 3.

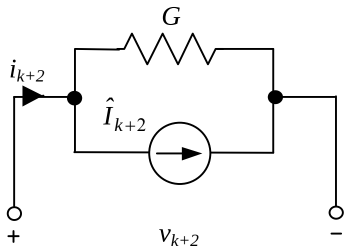

Fig. 3. Discrete model of the linear capacitor.

Similarly, the discrete models of inductors can be formed. For a nonlinear inductor (see Fig. 4) represented by

$$\phi = \hat{\phi}(i), \qquad v = \frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{15}$$

we create the discrete model shown in Fig. 5, described by the equation

$$v_{k+2} = r(i_{k+2}) + V_{k+2} \tag{16}$$

where

$$r(i_{k+2}) = a_1(h_1, h_2, \theta)\hat{\phi}(i_{k+2}) \tag{17}$$

$$\begin{aligned} V_{k+2} &= a_2(h_1, h_2, \theta)\hat{\phi}(i_{k+1}) + a_3(h_1, h_2, \theta)\hat{\phi}(i_k) + \\ &+ b(h_1, h_2, \theta)v_{k+1}. \end{aligned} \tag{18}$$

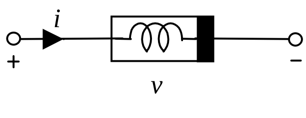

Fig. 4. A nonlinear inductor.

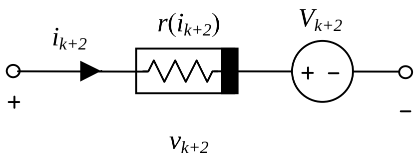

Fig. 5. Discrete model of the nonlinear inductor.

If the inductor is linear $(\phi = Li)$, then (16) becomes the linear equation

$$V_{k+2} = Ri_{k+2} + \hat{V}_{k+2} \tag{19}$$

$$R = a_1(h_1, h_2, \theta)L \tag{20}$$

$$\hat{V}_{k+2} = a_2(h_1, h_2, \theta)Li_{k+1} + a_3(h_1, h_2, \theta)Li_k + b(h_1, h_2, \theta)v_{k+1} \,. \tag{21}$$

Equation (19) enables us to form the discrete model of the linear inductor (see Fig. 6).

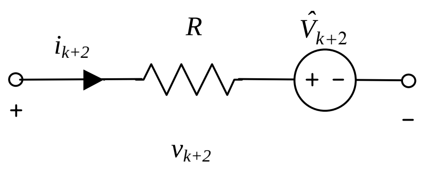

Fig. 6. Discrete model of the linear inductor.

Applying the discrete models of capacitors and inductors we transform the dynamic circuit into a sequence of resistive circuits. The resistive circuits are described by node equations and analyzed using the Newton-Raphson algorithm. In any step the parameters of the models are updated according to the equations (9)-(21) and the strategy described in Section I.

III. Numerical examples

To illustrate the algorithm developed in this paper we consider two numerical examples.

Example 1

Let us consider the univibrator shown in Fig. 7 [10] including the operational amplifier represented by the Chua-Lin model [5] with a single pole. We wish to find the voltage $v_3(t)$ within the time interval $[0,10]\,\mathrm{ms}$, with the zero initial conditions. Let the initial, minimal and maximal step sizes be 40μs, 10μs, 100μs, respectively. The initial values of θ is assumed to be zero.

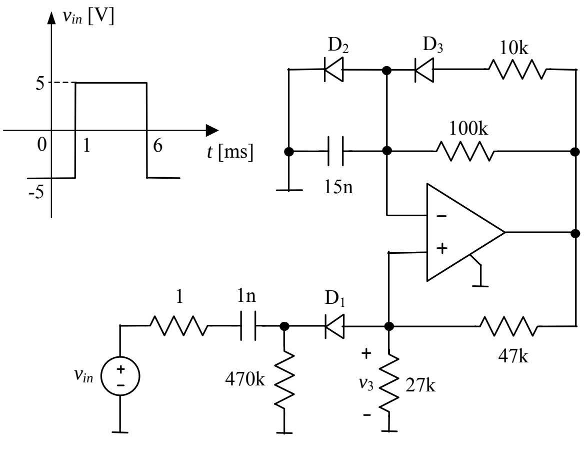

Fig. 7. The univibrator.

A part of the transient solution provided by the trapezoidal method, exhibiting spurious oscillations, is shown in Fig. 8.

The total number of the time steps is equal to 955 and the total number of the Newton-Raphson iterations is equal to 3032.

The same part of the transient solution obtained using the proposed method is shown in Fig. 9. The total number of the time steps is equal to 285, whereas the total number of the Newton-Raphson iterations is equal to 762.

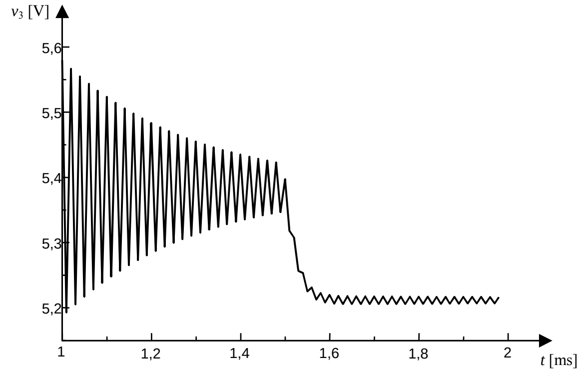

Fig. 8. A part of the transient solution of the univibrator provided by the trapezoidal method.

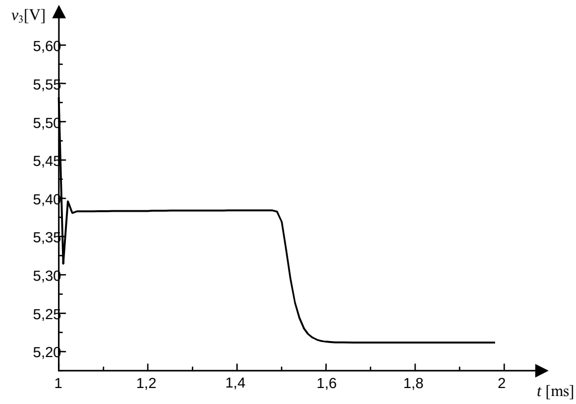

Fig. 9. The same part of the transient solution as in Fig. 8 obtained using the proposed method.

In this case the spurious oscillations are significantly reduced and immediately damped. Gear's method gives the transient solution similar to the solution obtained by the proposed method but it requires larger number of the time steps and the Newton-Raphson iterations.

Example 2

Figure 10 shows a linear waveform generator [11] containing a diode and four transistors represented by the Ebers-Moll models. We want to find the voltage $v(t)$ in the time interval $[0, 200]\,\mathrm{ns}$, with the zero initial conditions. Let the initial, minimal, and maximal step size be 2ns, 400ps, 4ns, respectively. The initial value of θ is assumed to be zero.

A part of the transient solution provided by the trapezoidal method, exhibiting spurious oscillations, is shown in Fig. 11. The total number of the time steps is equal to 316 and the total number of the Newton-Raphson iterations is equal to 1331.

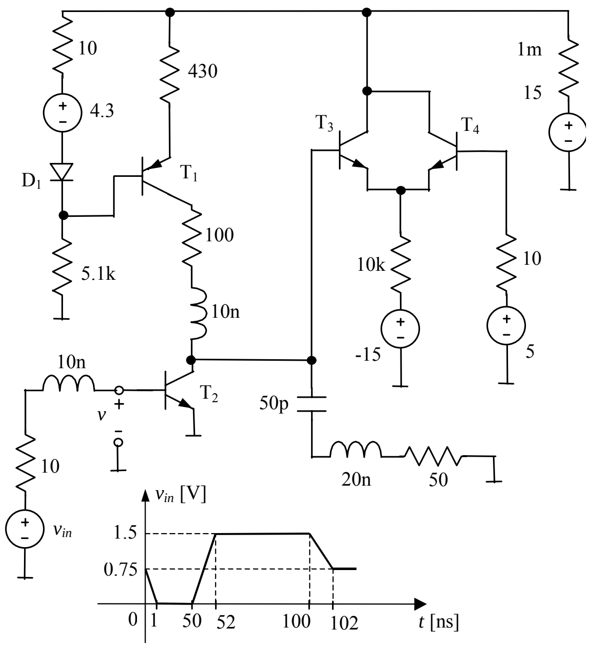

Fig. 10. The linear waveform generator.

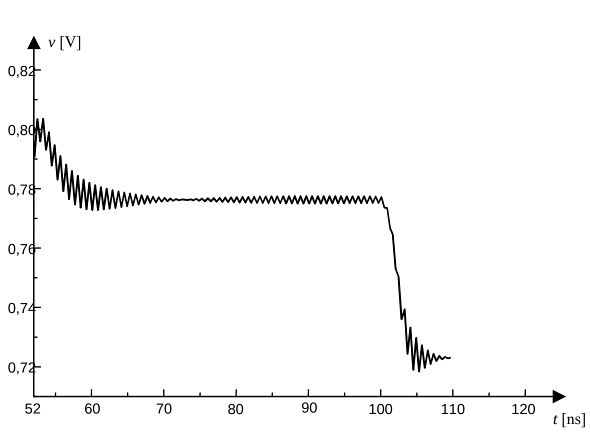

Fig. 11. A part of the transient solution of the linear waveform generator provided by the trapezoidal method.

The same part of the transient solution obtained using the proposed method is shown in Fig. 12. The total number of the time steps is equal to 245, whereas the total number of the Newton-Raphson iterations is equal to 904.

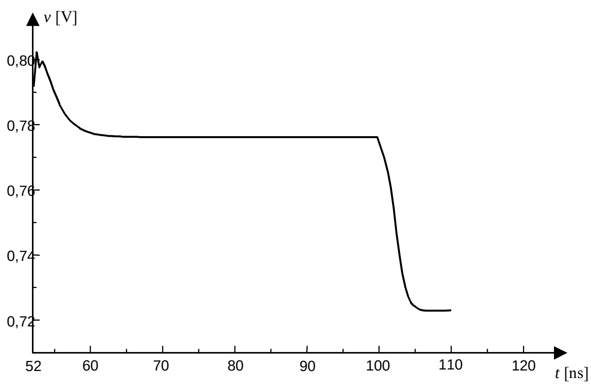

Fig. 12. The same part of the transient solution as in Fig. 11 obtained using the proposed method.

Similarly as in example 1 the spurious oscillations are significantly reduced and immediately damped. Gear's method gives the transient solution similar to the solution obtained by the proposed method but it requires larger number of time steps and the Newton-Raphson iterations.

IV. Conclusions

The described in this paper implementation of the proposed method, using the discrete models of capacitors and inductors, enables us to analyze a broad class of electronic circuits. Numerical examples confirm that the method is efficient and overcomes the main drawback of the trapezoidal scheme. In the case where the spurious oscillations are generated by the trapezoidal scheme the proposed method effectively damps them and leads to the solutions in shorter time. Furthermore, in all analyzed circuits the proposed method requires fewer time steps and the Newton-Raphson iterations than the Gear algorithm.

References

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