Effective Fourier Series-Less Method for Analysis of Periodic Non-Harmonic States in Linear Dynamical Systems and Hysteresis Loops of One-Period Energy

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***Abstract*–The paper presents a new approach to studies of periodic non-harmonic states in linear systems. It is completely independent on Fourier series method and is aimed on the application of a saw-tooth waveform to determining, on the line, the output-input relation in linear dynamical systems. Respective schemes for the unified representation of composite periodic non-harmonic waveforms are involved. The proposed method is simple and straightforward, the number of mathematical operations is minimized, and the structure of the load with a given supplying device or *vice versa* can be optimized according to proper desired level of one-period energy. The newly established method appears as a powerful broadly applicable technique to characterize non-harmonic periodic oscillations from a perspective different than that obtained by the method resulting from the Fourier series. Conditions leading to loops of the one-period energy are formulated and developed. Applying this newly recommended method leads to relinquishment of classic frequency analysis.**

Index Terms–**Periodic non-harmonic waveforms, Fourier series-less analysis, linear network, loops of one-period energy**

I. INTRODUCTION

IN RECENT YEARS, renewed attention has been directed toward the energy flow in dynamical systems, a problem which is denoted as one of the outstanding unsolved problems of modern technologies. In electric power systems the distortion of sinusoidal voltages and currents is one of the major energy quality concerns. The increased use of power electronic devices in the generation, transmission and utilisations of systems is accompanied by a corresponding growth of harmonic problems in power systems [1]. However, preventive measures are costly and their minimisation is becoming an important part of power system design, relying heavily on theoretical prediction. The traditional tools for analysis of nonsinusoidal waveforms are the Fourier transform and the sampling theorem of Shannon, Whittaker and Kotel'nikov [3].

In a general case of a linear system, under the assumption that a periodic steady-state exists, the output distorted waveform $f(t) = f(t+T)$ has a Fourier series representation

$$f(t) = F_0 + \sum_{n=1}^{\infty} F_n \cos(n\omega t + \psi_n) \tag{1}$$

where $\omega = 2\pi / T$ is the fundamental angular frequency, and constants F_0, F_n and ψ_n denote the *dc* value, amplitude and phase angle of the n^{th} harmonic of the waveform, respectively. It is easily seen that a periodic nonharmonic waveform can be expressed as a sum of infinite number of cosine components with multiple frequencies. For larger systems and complicated harmonic producing elements, more formal harmonic analysis methods are needed. This requires us to find the proper frame in which it is possible, in an easy way, to determine the periodic non-harmonic response of multivariable linear systems [2], [4], [6]. The time domain representation of a system by means of the concatenation procedure, as opposed to the frequency domain representation by means of the system transfer function, became the more advantageous approach to the representation of system dynamics.

The aim of this paper is to introduce a new Fourier series-less analysis method, based on applications of a saw tooth waveform for direct time-domain description, which analyzes the linear system voltage and current waveforms as much as more effective way in respect to the traditional frequency analysis. In addition, our new method can avoid the "singularity induced infinity" problem, which may happen at traditional analysis around singular points. Moreover, we present hysteresis loops for one-period energy obtained on the energy phase plane when the system is subjected to cyclic excitations. Illustration examples are emphasized using Matlab *mfiles* [5].

II. CARRYING SAW-TOOTH SIGNAL AND OTHER USEFUL WAVEFORMS

Basic waveforms such as the sine, saw-tooth, square and triangle still play an important role in today's signal processing by power electronic applications. These waveforms are referred to as "canonical" because of the significant role they play in Fourier series-less technique appearing as very effective in establishing a new procedure for studies of periodic nonharmonic states in linear systems. The square wave belongs to a larger class of waveforms called pulse waves with essentially two states: *on* or *off*. A saw-

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tooth wave *s(t)* is determined as

$$s(t) = s(t+T) = \frac{T}{2} - \frac{T}{\pi} arctan(cot\frac{\pi}{T}t) \qquad (2)$$

where $T = 2\pi$ denotes the period. The direct plot of (2) is shown in Fig.1. It will be used in the sequel as a *carrying periodic waveform* because of its linear growth during the period and repeated changes at regular intervals. In what follows we will use only *s* instead of *s(t)*.

To show a creative application of this waveform, some interesting periodic waveforms will be generated using relationships derived from their non-periodic time-domain origins.

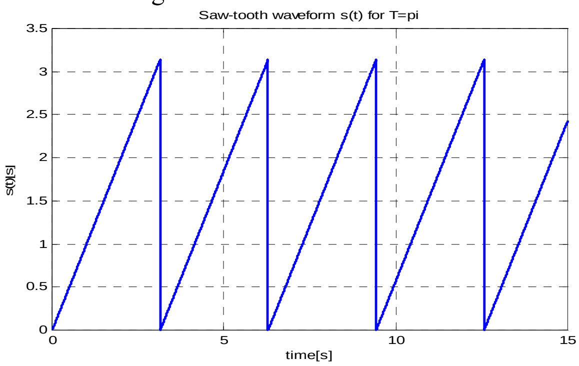

Fig.1. Saw-tooth waveform for $T=\pi$

In the sequel we will take advantages of such useful waveform as:

- relay function $r(t,\tau)$ called also *jump function* which can be defined by

$$r(t,\tau) = \frac{Abs(t-\tau)}{t-\tau} = \frac{t-\tau}{Abs(t-\tau)} \qquad (3)$$

and its periodic counterpart $r_T(t,\tau) = r_T(t+T,\tau)$ depending on *s*, i.e.

$$r_T(t,\tau) = r(s,\tau) \qquad (4)$$

The plot of (4) for $T = 2\pi$ and $\tau =$ is presented in Fig.2.

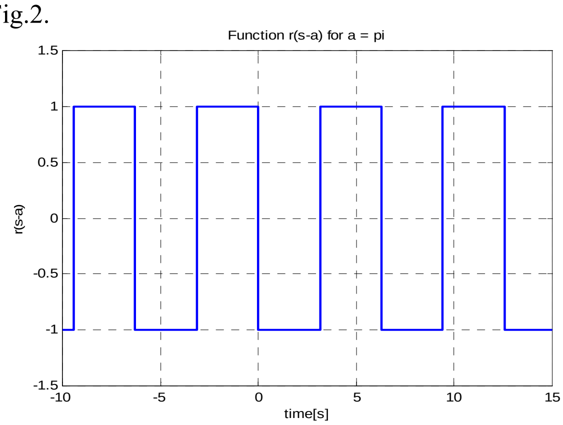

Fig.2. Jump waveform $r_T(t,\tau)$ for $T = 2\pi$ and $\tau = \pi$

Further, in a quite similar manner we can introduce waveforms: *switch-off function* denoted by *s-off(t, τ)* and determined as follows

$$s-off(t,\tau) = \begin{cases} -1 & \text{for t } \le \tau \\ 0 & \text{for t } \ge \tau \end{cases} \qquad (5)$$

switch-on function denoted by *s-on(t, τ)* and defined by

$$s-on(t,\tau) = \begin{cases} 0 & \text{for t } \le \tau \\ 1 & \text{for t } \ge \tau \end{cases} \qquad (6)$$

By the same way we can define many others waveforms which are very useful in analysis of non-harmonic periodic states of linear systems. It is worth to mention that for an adequate choice of the canonical waveforms and their arguments we can decompose complex signals with respect to *s*.

III. DISCONTINUOUS WAVEFORMS AND THEIR REPRESENTATIONS

In practice, the main sources of distortions of sinusoidal current and voltage waveforms are power electronic devices, which exercise controllability by means of multiple switching events within the fundamental frequency waveform. Quite obviously, if the source waveform is subject to jump changes the linear smoothing procedure is not a good choice anymore, because all linear networks confuse and remove the high frequency components from the output waveforms. For this reason, when a source waveform with jumps is applied to a linear network it causes a typical effect of "edge blurring" [3], [4]. The extraneous peaks in the square wave's Fourier series never disappear; they occur whenever the waveform is discontinuous, and will always be present whenever the signal has jumps [6]. We remark that the latter situation will be relevant in the present analysis. But blowing up the "fine structure" of the singular perturbation problem becomes visible and the full problem can be analyzed by applying or adapting standard methods from dynamical systems theory [8].

For the construction of an appropriate algorithm we consider a general setting, i.e., waveforms with discontinuities. We use the results known in the general statement for the special case of the concatenation and show that this leads to an elegant procedure in Fourier series-less analysis also in the situation of real jumps in the input as well as output waveforms.

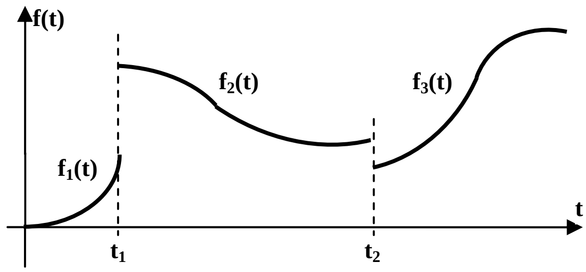

Fig.3. A waveform with jumps at $t = t_1$ and $t = t_2$

To cope with these effects we will describe discontinuous waveforms by using the saw-tooth waveform and its relatives such as switch-on and switch-of waveforms. Thus the wave-form shown in Fig.3 can be represented as

$$f(t) = f_1(t) + s-on(t,t_1)[f_2(t) - f_1(t)] + s-on(t,t_2)[f_3(t) - f_2(t)] \qquad (7)$$

where the waveform *s-on(t,t_k)*, *k =1, 2,* is determined by (6).

This intuitively appealing "switching rule" can be exploited in several ways. Using the switching approach (7) suggests the incorporation of a true smoothing element into the competition. For instance, in the case of the signal shown in Fig.2 the period equals $T = 2\pi$, and the discontinuity appears at $t_1 = \pi$. Prior to discontinuity the signal takes constant value equal to *-1* and post discontinuity it equals *1*. Thus applying (7) yields

$$\begin{aligned} r_T(t,\tau) &= -1 + s - on_{2\pi}(s,\pi)(1-(-1)) \\ &= -1 + 2s - on_{2\pi}(s,\pi) \end{aligned} \quad (8)$$

These calculations demonstrate how the signal jumps can be taken into account to a governing equation of a given linear system under periodic nonharmonic regime. The existence of solutions to such problems is discussed by making use the continuity and periodicity properties of appropriate waveforms in a given network. The idea is to find suitable expressions for the output waveforms in terms of steady state and transient components of solutions of differential equations describing instantaneous state of the network during particular intervals over one period of the bounded forcing term. Details in this direction we shall present in the next section.

IV. SOLUTIONS OF GOVERNING EQUATIONS FOR PERIODIC NON-HARMONIC STATES

Conditions associated with the analysis of complex waveforms including periodicity and continuity will be involved in this Section. In order to present a general algorithm let us consider steady state oscillations in the network shown in Fig.4. It corresponds to a single-phase bridge voltage source inverter *(VSI)* supplying an *RLC* load [7].

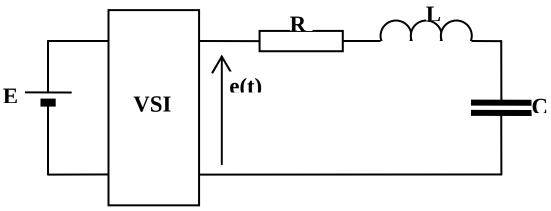

Fig.4. A network with a voltage source inverter *VSI*

$R = 2\Omega$, $L = 1H$, $C = 0.25F$, $E = 0.2V$, $e(t)=0.2r(t)V$.

The mathematical model of such a configuration takes the form

$$\ddot{z}(t) + 2\dot{z}(t) + 4z(t) = u(t) \quad (9)$$

where the forcing term $u(t) = 0.8\ r(t)$ is presented in Fig.5. The amplitude of this waveform was scaled by 50%.

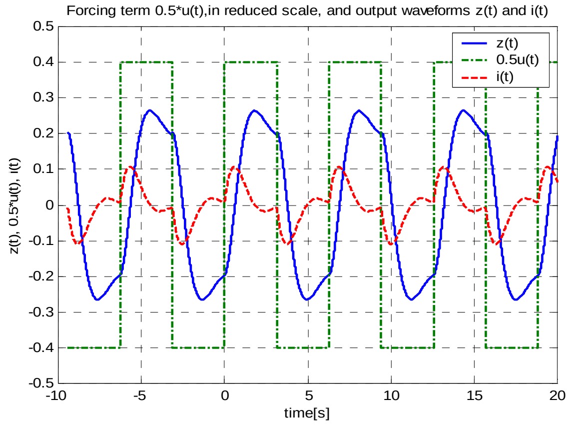

Fig. 5. The input and output waveforms

In this case the self-frequencies of the network are as follows

$$p_1 = -1 + j\sqrt{3}, \qquad p_2 = -1 - j\sqrt{3} \quad (10)$$

The steady-state solutions for the periodic output waveform in successive semi-periods of the input wave take the forms

- for $0 \le t \le \pi$

$$z_1(t) = 0.2 + e^{-t}[G_1\cos(\sqrt{3}t) + H_1\sin(\sqrt{3}t)] \quad (11)$$

- for $\pi \le t \le 2\pi$

$$z_2(t) = -0.2 + e^{-t}[G_2\cos(\sqrt{3}t) + H_2\sin(\sqrt{3}t)]$$

(12)

where the integration constants G_1, H_1, G_2 and H_2 are to be determined from the respective conditions for the periodicity and analytical continuity of the total solution $z(t)$.

Taking into account the periodicity and analytical continuity of the output waveform gives

$$z_1(0) = z_2(2\pi),\ \dot{z}_1(0) = \dot{z}_2(2\pi)$$

and

$$z_1(\pi) = z_2(\pi),\ \dot{z}_1(\pi) = \dot{z}_2(\pi)$$

Solving the above equalities with respect to the integration constants yields

$G_1 = -0.3953$, $G_2 = 9.9876$, $H_1 = -0.2117$, $H_2 = -3.6316$

Substituting these values into (11) and (12) and mapping the solution of (9) into the *s* domain gives

$$\begin{aligned} z(t) &= z_1(s) + [z_2(s) - z_1(s)]s - on(s) \\ &= -0.2r(s) + e^{-s}\{[-0.3953 + 10.3882s - on(s)]\cos(\sqrt{3}s) \\ &\quad + [-0.2117 + (-3.4119son(s)]\sin(\sqrt{3}s)\} \\ &= -0.2r(s) + e^{-s}\{[4.7988 + 5.1941r_T(s)]\cos(\sqrt{3}s) \\ &\quad + [-1.9177 - 1.706r(s)]\sin(\sqrt{3}s)\} \end{aligned}$$

(13)

Fig.5 represents the forcing waveform *u(t)*, in twice reduced scale, and the output waveforms of the capacitive voltage *z(t)* and load current *i(t)*. It is worth pointing out that the shapes of three plots shown in Fig. 5 differ importantly one to other but the period of the output waveforms is the same as that of the input waveform. Essentially, the peaks and troughs of the output current *i(t)*correspond to maximum and minimum slopes in time of the output voltage *u(t)*, respectively. Note also that at $t = T/2$ the current in the load is negative and it may be possible to rely on load commutations of inverter thyristors. This problem is beyond the scope of the present paper.

V. HYSTERESIS LOOPS OF ONE-PERIOD ENERGY

In many physical applications of periodic waveforms the interest lies in interpretable basis representations for the generated or consumed energy during an interval of the activity of an energy source or a load, respectively. As is well known, all up-to-date used methods for energy determination of a system element working in a periodic non-harmonic steady state have many insufficiencies which vary from one case to other and importantly depend on the assumed interpretation of particular components of the apparent power defined in the complex number domain [2], [4], [7]. To avoid this problem we consider a much more general setting, i.e., hysteresis loops of one-period energy on an energy phase plane.

We present hysteresis loops in electric charge-voltage or, equivalently, magnetic flux-current curves obtained from numerical simulations of the steady state energy when the system is subjected to periodic excitations. We use the results known in a general setting for special cases of the system elements and show that this leads to an elegant algorithm in Fourier series-less analysis also in the case of real periodic nonharmonic waveforms.

The steady state energy $W(\Delta t)$ delivered by the source $v(t)$ to its load during a time interval $\Delta t = nT$, where $n >> 0$ denotes a positive integer, is expressed by

$$W(\Delta t) = nW_T \qquad (14)$$

where W_T denotes the energy delivered to the load during one period of the input and output waveforms. Thus in the periodic non-harmonic state it is sufficient to evaluate W_T and then multiplying it by n yields the energy delivered or consumed by a network element during the given time interval Δt.

The derivation of the corresponding expression for W_T gives

$$\begin{aligned} W_T &= \int_0^T v(t)i(t)dt = \int_0^T v(t)\frac{d}{dt}\left(\int i(\tau)d\tau\right)dt \\ &= \int_{q(0)}^{q(T)} v(t)dq(t) = \int_{\Psi(0)}^{\Psi(T)} i(t)d\psi(t) \end{aligned} \qquad (15)$$

where

$$q(t) = \int i(t)dt \quad \text{and} \quad \Psi(t) = \int u(t)dt \qquad (16)$$

denote the electric charge and magnetic flux, respectively.

It follows from expression (15) that the area enclosed by a loop on the energy phase plane with coordinates $(q(t),\ v(t))$ or, equivalently, $(\psi(t), i(t))$ determines the one-period energy W_T delivered to, or consumed by, respectively, a one-port network being under periodic non-harmonic conditions.

For the network shown in Fig. 4 the one-period energy W_T delivered by the source to its load results of calculations presented in the previous Section. If we take the input voltage $e(t)$ as one coordinate of the energy phase plane then the choice of the electric charge $q(t) = \int i(t)dt = Cz(t)$ as the second coordinate of the energy phase plane is implied. With these coordinates we can draw the loop of the one-period energy W_T of the source supplying the RLC load. It is presented in Fig.6. The important point to note here is the exceptional form of the one-period energy loop. A rectangular hysteresis loop is obtained after a source voltage reversal between plus and minus E. Its area can be very easy determined and in result we have $W_T \cong 0.0392$ J for $C_1 = 0.4$ F but $W_T \cong 0.0174$ J for $C_2 = 0.1$ F with all other unchanged parameters. Thus, this area varies with capacitance of the load and with the frequency $f = 1/T$ of the forcing term. The overall shape of the hysteresis loop is in quantitative agreement with experimental results, much more so than in the model with Fourier series analysis.

In an another case of analysis when the losses of the capacitor are taken into account the corresponding loops of one-period energies are shown in Fig.7. They are determined for three values of the capacitance with all other parameters unchanged. One of the advantages of introducing this facility into the system studies is the ability to combine useful properties in the time domain of each of the structure elements.

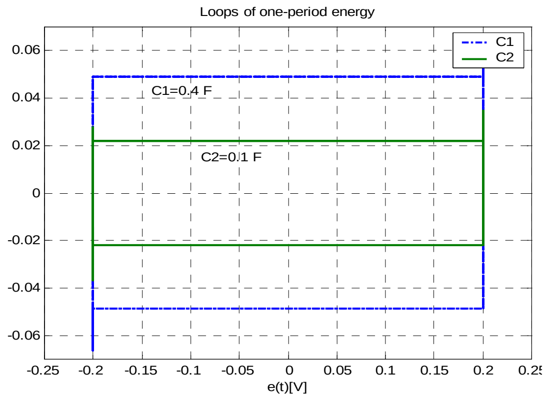

Fig.6. Loops of one-period energies of the source in Fig.5

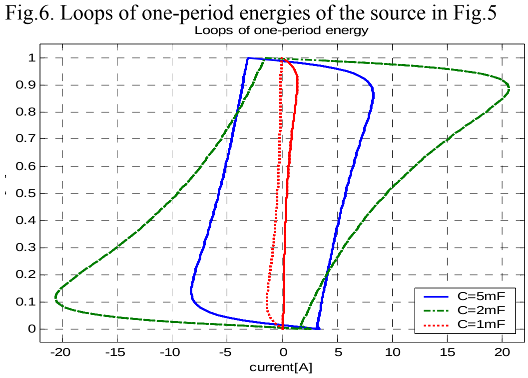

Fig.7. Loops of one=period energies when losses are added to the capacitor

VI. FINAL REMARKS AND CONCLUSIONS

We have discussed properties of linear dynamical systems with dissipation and non-harmonic periodic waveforms. A new systematic method of dynamical network analysis without use of Fourier series approaches has been presented. It can serve as a foundation for the derivation of hysteresis loops for the one-period energy of linear network components. A common feature of these loops is that all problems concerning the non-harmonic periodic states in linear electrical networks can be reported to studies of the somewhat complementary aspect of shaping the energy of the network in the energy phase plane.

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