Comparative Study of Jacobian Calculation Techniques in Electrical Impedance Tomography

Lubomír Brančík

***Abstract* — The electrical impedance tomography is a means how to localize inhomogeneities in some bordered homogenous medium when the object under investigation is accessible to the measuring only at specific points of its boundary. In the method various optimization techniques are in use. For their successful application it is useful to determine Jacobian which helps to accelerate an iterative process. In the paper two ways of Jacobian calculation are compared above all from the point of view of the CPU time. Namely, the first method is based on the reciprocity theorem application, the second one then on the direct differentiation of the nodal-analysis matrix equation. Both methods utilize a precomputed properly chosen finite-element mesh, and has been programmed in the Matlab language environment.**

I. Introduction

Consider a two-dimensional region characterized by a conductivity γ, which is discretized using the finite element method [1,2]. Then its boundary contains the relevant number of nodes, when some of them can be used as inputs for driving and the others as outputs for measuring, see Fig.1.

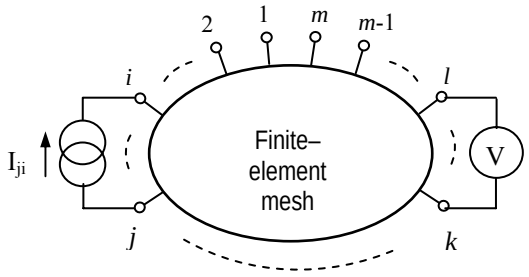

Fig. 1 Selected outer input/output nodes

The number of upper-most nodes (terminals) chosen for testing process is indicated by *m*, the input by the pair *i*-*j*, and the output by the pair *k*-*l*. There are many various possibilities how to choose the pairs of these terminals, and it strong depends on the problem under consideration. For our goal it does not matter which ones are chosen, we will just assume that *M* different variants of the input/output terminals are somehow set. Neither a form of the finite-element mesh will play an essential role in the CPU times comparison. We will just consider that the region is divided into *N* finite elements. With the above designations an assesment of Jacobian means to compute the matrix **J** of the order $M \times N$, where its particular elements are equal to the absolute sensitivities $\partial V_s / \partial \gamma_r$, where $s = 1, 2, \cdots M$, and $r = 1, 2, \cdots N$. It means the sensitivities of output voltages corresponding to all the input/output variants to changes of the conductivity of every finite element are covered by the Jacobian.

From point of view of the circuit theory the system can be considered as the passive linear multiterminal network since it is composed only of certain number of appropriate linear conductances. In both methods the nodal analysis matrix equation is used

$$\mathbf{GV} = \mathbf{I} \ , \qquad (1)$$

where **G** is a conductance matrix, **V** is a column vector of the nodal voltages, and **I** is a column vector of the currents reflecting independent current sources. The reference node is always advantageously set inside the region. However, the equation (1) will be treated by different way according to chosen method as follows.

II. Reciprocity theorem application

At first we show how the reciprocity theorem can be utilized for the absolute sensitivity calculation. Choose two pairs of external nodes, *a*–*b* as the input and *c*–*d* as the output, see Fig. 2 [3]. The aim is to calculate the absolute sensitivity of the output voltage V_{cd} to the change of a certain conductivity G_k, i. e. the value $\partial V_{cd} / \partial G_k$, when the input is supplied from the current source I_{ba}. Consider further the situation when the positions of supplying and measuring are interchanged.

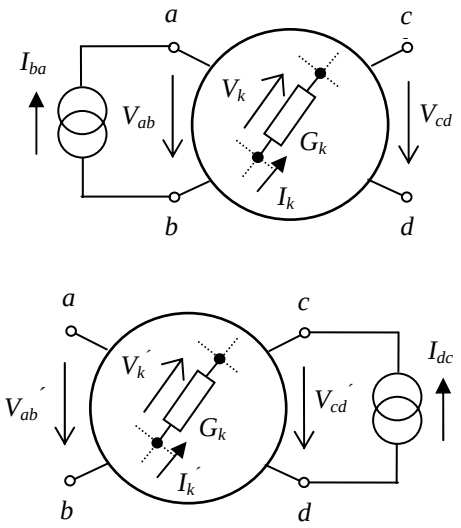

Fig. 2. Problem of sensitivity calculation (part 1)

Suppose the region containes *n* conductances. Then considering Tellegen's theorem we get for the powers from Fig. 2 the formulae

Author is with the Institute of Theoretical and Experimental Electrical Engineering, Faculty of Electrical Engineering and Communication, Brno University of Technology, Purkyňova 118, 612 00 Brno, Czech Republic, e-mail: *brancik@feec.vutbr.cz*

$$V_{ab}I_{ba} = \sum_{j=1}^{n} V_j I_j \quad \text{(2a)} \quad , \quad V'_{cd}I_{dc} = \sum_{j=1}^{n} V'_j I'_j \ . \quad \text{(2b)}$$

Let current sources I_{ba} and I_{dc} act simultaneously at both pairs of the terminals, see Fig. 3.

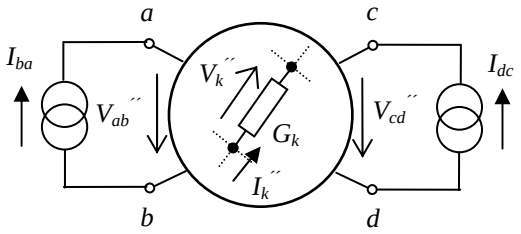

Fig. 3. Problem of sensitivity calculation (part 2)

Considering again Tellegen's theorem we have

$$V''_{ab}I_{ba} + V''_{cd}I_{dc} = \sum_{j=1}^{n} V''_j I''_j \quad . \quad (3)$$

Now taking linearity of the system into account, and using superposition theorem, we can write

$$(V_{ab}+V'_{ab})I_{ba} + (V_{cd}+V'_{cd})I_{dc} = \sum_{j=1}^{n}(V_j+V'_j)(I_j+I'_j) . \quad (4)$$

Substituting (2a) and (2b) into (4) we get after simple arrangement

$$V'_{ab}I_{ba} + V_{cd}I_{dc} = \sum_{j=1}^{n}(V_j I'_j + V'_j I_j) = 2\sum_{j=1}^{n}\frac{I_j I'_j}{G_j} . \quad (5)$$

Now when defining

$$I_{ba} = I_{dc} = I \quad (6)$$

then the equality

$$V'_{ab} = V_{cd} \quad (7)$$

implies of the reciprocity theorem. Then (5) leads to

$$V_{cd} = \frac{1}{I}\sum_{j=1}^{n}\frac{I_j I'_j}{G_j} \ . \quad (8)$$

Finally the absolute sensitivity in demand is got as

$$\frac{\partial V_{cd}}{\partial G_k} = -\frac{1}{I}\frac{I_k I'_k}{G_k^2} = -\frac{1}{I}V_k V'_k \ . \quad (9)$$

Now consider the *r*–th finite element consists of *R* conductances equal to

$$G_k^{(r)} = \gamma_r K_k^{(r)} \ , \ k = 1,2,\cdots,R \ , \quad (10)$$

where $K_k^{(r)}$ are parameters resulting from the finite element method. Then the whole change of the output voltage is given by the total differential

$$dV_{cd} = \sum_{k=1}^{R}\frac{\partial V_{cd}}{\partial G_k^{(r)}}dG_k^{(r)} = d\gamma_r \sum_{k=1}^{R}\frac{\partial V_{cd}}{\partial G_k^{(r)}}K_k^{(r)} \quad (11)$$

and, therefore

$$\frac{\partial V_{cd}}{\partial \gamma_r} = -\frac{1}{I}\sum_{k=1}^{R} V_k^{(r)} V_k'^{(r)} K_k^{(r)} \ , \quad (12)$$

where (9) was taken into account.

Note:

The formula (12) is an analog to the relation which was derived in [4] from the point of view of the field theory for the continuous case, namely

$$\frac{\partial V_{cd}}{\partial \gamma_r} = -\frac{1}{I}\int_{\Omega_r} \nabla V_\Phi^r \cdot \nabla V_\Psi^r \, d\Omega \ , \quad (13)$$

where V_Φ^r and V_Ψ^r are potential distributions related to the surface element Ω_r, generated due to acting a current *I* on the *a*–*b* and *c*–*d* terminals, respectively, γ_r is its conductivity, and ∇ is the nabla operator.

Thus for the Jacobian computation it is necessary to solve not only the equation (1) leading to $V_k^{(r)}$ values but also another one resulting in $V_k'^{(r)}$ determination. Considering (10) we can write these equations as

$$\gamma\mathbf{K}\mathbf{V} = \mathbf{I} \quad \text{(14a)} \quad , \quad \gamma\mathbf{K}\mathbf{V}' = \mathbf{I}' \ , \quad \text{(14b)}$$

where **K** is the system matrix resulting from the finite element method. As the matrix is very sparse this fact should be taken into account not to vaste the computer memory and to save the CPU time. In Matlab language both equations (14) can be solved in parallel using the Gaussian elimination method very effectively, taking them directly in sparse forms, as (in Matlab notation)

$$\widetilde{\mathbf{V}} = \gamma^{-1}\mathbf{K} \setminus \widetilde{\mathbf{I}} \quad , \quad (15)$$

where $\widetilde{\mathbf{V}} = [\mathbf{V}, \mathbf{V}']$ and $\widetilde{\mathbf{I}} = [\mathbf{I}, \mathbf{I}']$ matrices consist of the original vectors as their columns. Moreover, the right side of (15) can contain not only two but all the necessary current vectors which are needed for the Jacobian $M \times N$ matrix assesment, formally written

$$\widetilde{\mathbf{V}}_J = \gamma^{-1}\mathbf{K} \setminus \widetilde{\mathbf{I}}_J \ . \quad (16)$$

Finally from the particular columns of the matrix $\widetilde{\mathbf{V}}_J$ (nodal voltage vectors) the necessary branch voltages are determined being then able to use (12). These are already simple little CPU-time consuming operations.

III. NODAL ANALYSIS EQUATION DIFFERENTIATION

An alternative method of the Jacobian calculation lies in the direct differentiation of the nodal analysis matrix equation (1). Thus following it we can write

$$\frac{\partial}{\partial \gamma_r}(\mathbf{G}\mathbf{V}) = \frac{\partial}{\partial \gamma_r}\mathbf{I} = \mathbf{0} \ , \quad (17)$$

where **0** is a zero column vector. This is due to the fact that the vector **I** contains invariables (currents of independent sources). The (17) will proceed at

$$\frac{\partial \mathbf{G}}{\partial \gamma_r}\mathbf{V} + \mathbf{G}\frac{\partial \mathbf{V}}{\partial \gamma_r} = \mathbf{0} \quad (18)$$

and, finally

$$\frac{\partial \mathbf{V}}{\partial \gamma_r} = -\mathbf{G}^{-1}\frac{\partial \mathbf{G}}{\partial \gamma_r}\mathbf{G}^{-1}\mathbf{I} \quad , \tag{19}$$

where $\mathbf{V} = \mathbf{G}^{-1}\mathbf{I}$ was substituted according to (1). This formula corresponds to one used as well in [5]. Because of (10) the $\partial\mathbf{G}/\partial\gamma_r$ matrix in (19) contains altogether zero elements with the exception of those that belong to the *r*–th finite element. Thus this differentiation leads to the $\partial G_k^{(r)}/\partial\gamma_k = K_k^{(r)}$ terms at relevant places of the matrix. Besides, if the equation $\mathbf{G}^{-1} = \gamma^{-1}\mathbf{K}^{-1}$ is considered, the (19) results in

$$\frac{\partial \mathbf{V}}{\partial \gamma_r} = -\gamma^{-2}\mathbf{K}^{-1}\mathbf{K}^{(r)}\mathbf{K}^{-1}\mathbf{I} \ , \tag{20}$$

with $\mathbf{K}^{(r)}$ as a finite–element system matrix reduced as explained above.

To compute the Jacobian $M \times N$ matrix effectively it is again possible to replace the current vector **I** by the matrix $\mathbf{I}_J$ containing all the necessary input current vectors which are needed for this calculation. Moreover, because of the necessity to assemble the reduced matrix $\mathbf{K}^{(r)}$ repeatedly for all the finite elements (i. e. *N* times), the further procedure will be portioned into two steps as follows. Firstly, in Matlab notation, the matrix

$$\mathbf{V}_J = \gamma^{-1}\mathbf{K} \setminus \mathbf{I}_J \tag{21}$$

is computed only once by the Gaussian elimination method. Then for each finite element the matrix of particular vectors of nodal voltage sensitivities is got

$$\frac{\partial \mathbf{V}_J}{\partial \gamma_r} = -\gamma^{-1}(\mathbf{K} \setminus \mathbf{K}^{(r)})\mathbf{V}_J \ . \tag{22}$$

Herein the Gaussian elimination method is again used to solve the system in the parenthesis in parallel. The (22) is solved repeatedly for all $r = 1, 2, \cdots N$. All the above operations run again on the sparse matrices.

Now the absolute sensitivity of a considered output voltage can be obtained as

$$\frac{\partial V_{cd}}{\partial \gamma_r} = \frac{\partial}{\partial \gamma_r}\left(V_c - V_d\right) = \frac{\partial V_c}{\partial \gamma_r} - \frac{\partial V_d}{\partial \gamma_r} \quad , \tag{23}$$

where the particular nodal voltages sensitivities occur on the right side. These can already be got from (22) by simple little CPU-time consuming operations.

IV. Experimental methods comparison

A. Finite-element mesh description

To compare the above discussed methods consider a simple 2D circular region Ω with a unitary radius, discretized in the way as shown in Fig. 4. Generally the region is divided into *p* concentric zones of the size $1/p$ in a radial direction and *q* sectors of the angel $2\pi/q$ in a tangential direction [2,3]. The Fig. 4 shows the case for *p* = 4 and *q* = 12. Thus from the total number *N* = *pq* elements the *q*(*p*–1) are of the trapezoidal form, and the *q* in the most inner zone are of the triangular form. An ordinal number of element assignes simultaneously the ordinal number to a node related to the upper right corner. In Fig. 4 the uppermost nodes are numbered in this way. The reference node 0 is put in the centre of the region.

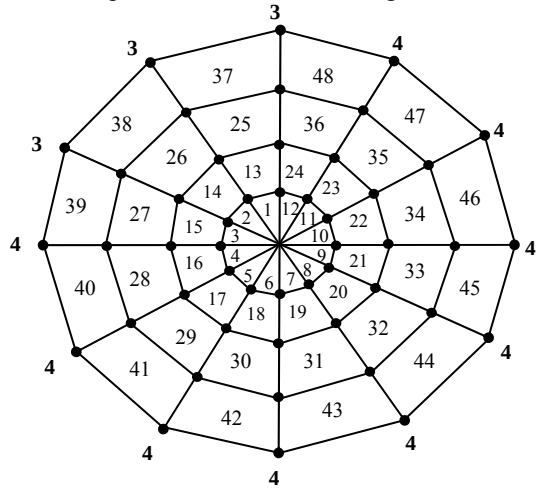

Fig. 4. Finite–element mesh under

In Fig. 5 there are possible *r*–th finite elements in detail. Nodal indices are correct with the exception $r = kq$, *k* integer. In this case the indices $r+1$ and $r-q+1$ must be replaced by the indices $r-q+1$ and $r-2q+1$, respectively. In Fig. 4 it is pertained to the elements 12, 24, 36 and 48.

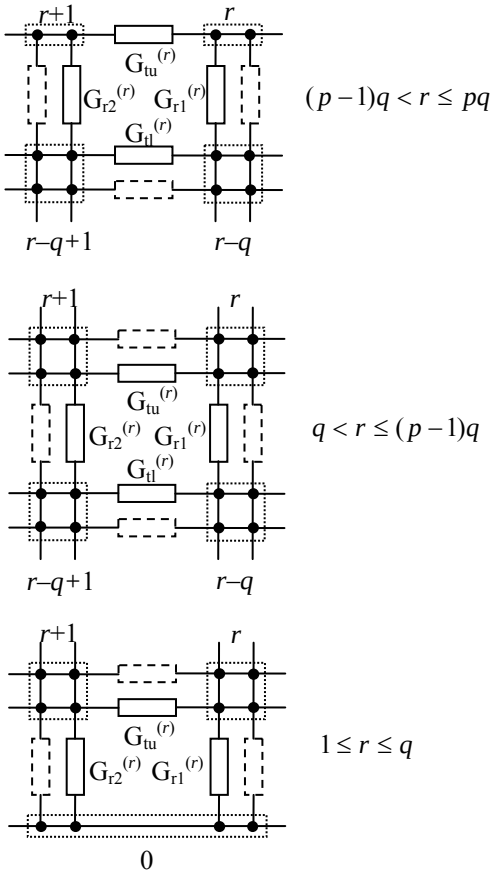

Fig. 5. Particular finite elements in detail

The inner three or four conductances belong to the *r*–th finite element (drawn by a solid line), the radial $G_r^{(r)}$, and the tangential lower $G_{tl}^{(r)}$ and upper $G_{tu}^{(r)}$.

Because of the centric symmetry the number of their different values is equal to the number of zones in radial direction *n*. In Matlab notation the zone index *i* can be got as $i = ceil(r/q)$ at first. Then values of the conductances are determined as

$$G_{r1}^{(i)} = G_{r2}^{(i)} = \gamma K_r^{(i)} \quad , \qquad i = 1,2,\ldots, p \quad , \tag{24}$$

$$G_{tu}^{(i)} = \gamma K_{tu}^{(i)} \quad , \qquad i = 1,2,\ldots, p \quad , \tag{25}$$

$$G_{tl}^{(i)} = \gamma K_{tl}^{(i)} \quad , \qquad i = 2,3,\ldots, p \quad , \tag{26}$$

where γ is the conductivity, and the finite-element parameters are given according to [2,3].

B. Jacobian calculation

The above chosen finite-element mesh leads to the implications for the particular methods as follows.

For the reciprocity theorem method the formula for the absolute sensitivity (12) results in

$$\frac{\partial V_{cd}}{\partial \gamma_r} = -\frac{1}{I}\left[\left(V_{r1}V'_{r1} + V_{r2}V'_{r2}\right)K_r + V_{tu}V'_{tu}K_{tu} + V_{tl}V'_{tl}K_{tl}\right]^{(r)}, \tag{27}$$

if $r > q$ is valid (two upper pics in Fig. 5), and when $R = 4$. Otherwise, in the case $R = 3$ (the bottom pic in Fig. 5), the last term in (27) is missing.

For the method of nodal-analysis matrix equation differentiation the reduced matrix $\mathbf{K}^{(r)}$ has the form

columns

$$\begin{array}{c} \\ \text{rows} \end{array} \begin{array}{c} \begin{array}{cccccccc} & r-q & & r-q+1 & & r & & r+1 \end{array} \\ \begin{array}{c} \\ r-q \\ r-q+1 \\ \\ r \\ r+1 \\ \\ \end{array} \begin{bmatrix} \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & K_r + K_{tl} & -K_{tl} & \cdots & -K_r & 0 & \cdots & \\ \cdots & -K_{tl} & K_r + K_{tl} & \cdots & 0 & -K_r & \cdots & \\ \vdots & & \vdots & & \vdots & & \vdots & \\ \cdots & -K_r & 0 & \cdots & K_r + K_{tu} & -K_{tu} & \cdots & \\ \cdots & 0 & -K_r & \cdots & -K_{tu} & K_r + K_{tu} & \cdots & \\ \vdots & & \vdots & & \vdots & & \vdots & \end{bmatrix} \end{array}$$

if $r > q$ is satisfied, that is only 12 elements in this matrix have nonzero values. Similarly, for $r \le q$, the reduced matrix has only 4 nonzero relevant elements, corresponding to the lower-right square block related to the indices *r* and *r*+1, as can be seen above.

For the Jacobian calculation we follow the Fig.1, when the number of terminals is chosen $m = 12$. The particular input/output variants are defined in Tab.1.

TABLE I. SETUP OF INPUT/OUTPUT PAIRS OF TERMINALS

Driving	Measuring
(1–2)	(2–3), (3–4),…, (12–1)
(2–3)	(3–4),…, (12–1)
⋮	⋮
(11–12)	(12–1)

Their total number is given by $M = m(m-1)/2 = 66$ [4]. The number of the finite elements varies from $N = 4 \times 12 = 48$ (this one corresponds to Fig. 4) to $N = 60 \times 180 = 10800$, namely in the series $N = pq$, where $p = 4k$ and $q = 12k$, with $k = 1,2,\cdots,15$. Such a choice enables spacing 12 driving/measuring uppermost terminals around the region always regularly.

Both discussed methods were programmed on the PC with Pentium IV 2GHz/256MB. The results are practically the same as they differ from about 10^{-22} to 10^{-21} in the sense of root-mean-square errors. The CPU time versus the finite-elements number relations are shown in logarithmic scales in Fig. 6.

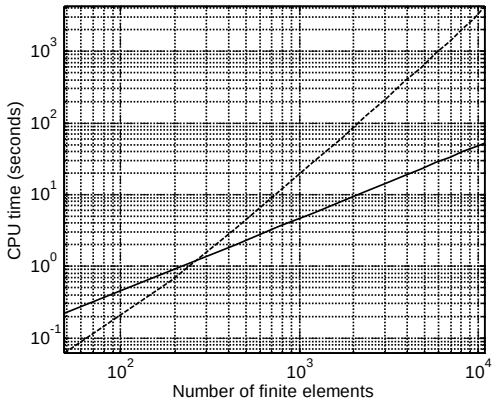

Fig. 6. CPU times comparison

A solid line corresponds to the reciprocity theorem method, a dashed line then to the method of the nodal -analysis equation differentiation. As can be seen for the numbers of the finite elements which are usually used in practice the method based on the reciprocity theorem is much faster. For example, for *N = 1200* it is about 5 times, for *N = 10800* already about 85 times. The contrary result is occuring only for small numbers of the finite elements, approximately under *N = 250*. Of course, these conclusions are depended on techniques of the programming, therefore they can somewhat differ of those gained by other prospective programmers. Moreover results will also depend on a chosen number *m* of terminals under testing.

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